All About Fuzzy Description Logics

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About Vagueness

On the Existence of Vague Concepts

What are vague concepts and do they exists?

- What are the pictures about?

- A concept is vague whenever its extension is deemed lacking in clarity
  - Aboutness of a picture or piece of text
  - Tall person
  - High temperature
  - Nice weather
  - Adventurous trip
  - Similar proof
- Vague concepts:
  - Are abundant in everyday speech and almost inevitable
  - Their meaning is often subjective and context dependent

On the Existence of Vague Objects

What are vague objects and do they exists?

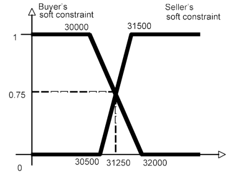
- Are there vague objects in the pictures?

- An object is vague whenever its identity is lacking in clarity
  - Dust
  - Cloud
  - Dunes
  - Sun
- Vague objects:
  - Are not identical to anything, except to themselves (reflexivity)
  - Are characterised by a vague identity relation (e.g. a similarity relation)
- BTW: example of *uncertain object*: “habitable Earth-like planet in universe"

Vague Statements

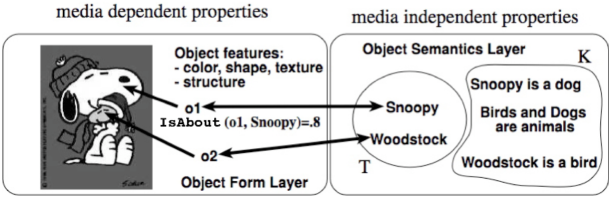
- A statement is vague whenever it involves vague concepts or vague objects
  - Heavy rain
  - Tall person
  - Hot temperature
- The truth of a vague statement is a matter of degree, as it is intrinsically difficult to establish whether the statement is entirely true or false
  - There are 33 °C. Is it hot?

Sources of Vagueness: Matchmaking



- A car seller sells an Audi TT for 31500 €, as from the catalog price.
- A buyer is looking for a sports-car, but wants to to pay not more than around 30000 €
- Classical DLs: the problem relies on the crisp conditions on price.
- More fine grained approach: to consider prices as vague constraints (fuzzy sets) (as usual in negotiation)
  - Seller would sell above 31500 €, but can go down to 30500 €
  - The buyer prefers to spend less than 30000 €, but can go up to 32000 €
  - Highest degree of matching is 0.75 . The car may be sold at 31250 €.

Sources of Vagueness: Multimedia information retrieval

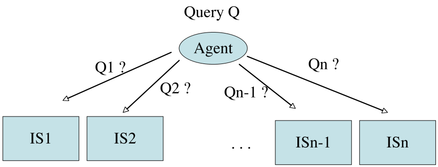


IsAbout		
ImageRegion	*Object ID*	*degree*
*o*1	*snoopy*	0.8
*o*2	*woodstock*	0.7
⋮	⋮	

"Find top-*k* image regions about animals"

$$Query(x) \leftarrow ImageRegion(x) \wedge isAbout(x, y) \wedge Animal(y)$$

Sources of Vagueness: Distributed Information Retrieval



Then the agent has to perform automatically the following steps:

1. The agent has to select a subset of relevant resources $\mathscr{S}' \subseteq \mathscr{S}$, as it is not reasonable to assume to access to and query all resources (resource selection/resource discovery);
2. For every selected source $\mathcal{S}_i \in \mathscr{S}'$ the agent has to reformulate its information need Q_A into the query language $\mathcal{L}_i$ provided by the resource (schema mapping/ontology alignment);
3. The results from the selected resources have to be merged together

Sources of Vagueness: Vague database query

HotelID	*hasLoc*
*h*1	*hl*1
*h*2	*hl*2
⋮	⋮

ConferenceID	*hasLoc*
*c*1	*cl*1
*c*2	*cl*2
⋮	⋮

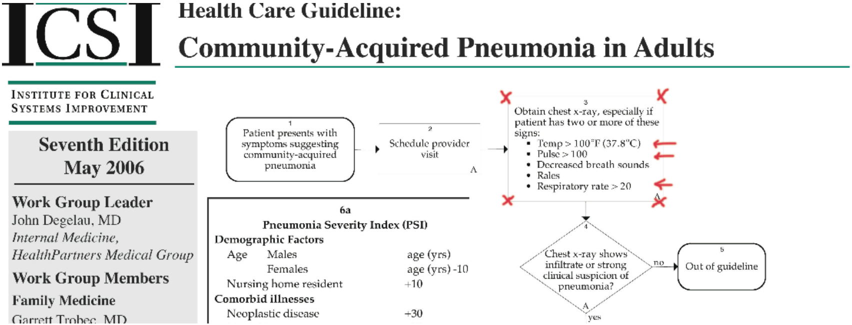
hasLoc	*hasLoc*	*distance*
*hl*1	*cl*1	300
*hl*1	*cl*2	500
*hl*2	*cl*1	750
*hl*2	*cl*2	800
⋮	⋮	

hasLoc	*hasLoc*	*close*	*cheap*
*hl*1	*cl*1	0.7	0.3
*hl*1	*cl*2	0.5	0.5
*hl*2	*cl*1	0.25	0.8
*hl*2	*cl*2	0.2	0.9
⋮	⋮	⋮	

"Find top-*k* cheapest hotels close to the train station"

$$q(h) \leftarrow hasLocation(h, hl) \land hasLocation(\texttt{train}, cl) \land close(hl, cl) \land cheap(h)$$

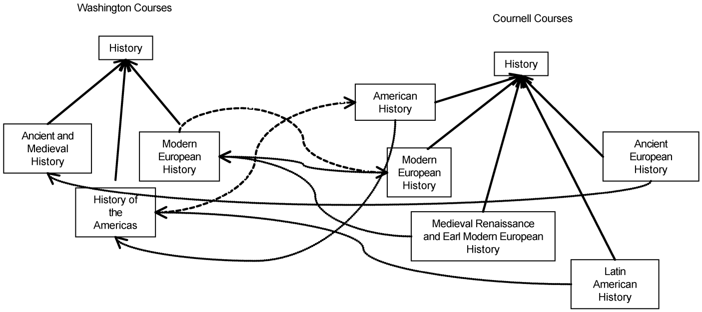
Sources of Vagueness: Health-care: diagnosis of pneumonia



- E.g., *Temp* = 37.5, *Pulse* = 98, *RespiratoryRate* = 18 are in "danger zone" already
- Temperature, Pulse and Respiratory rate: these constraints are rather vague

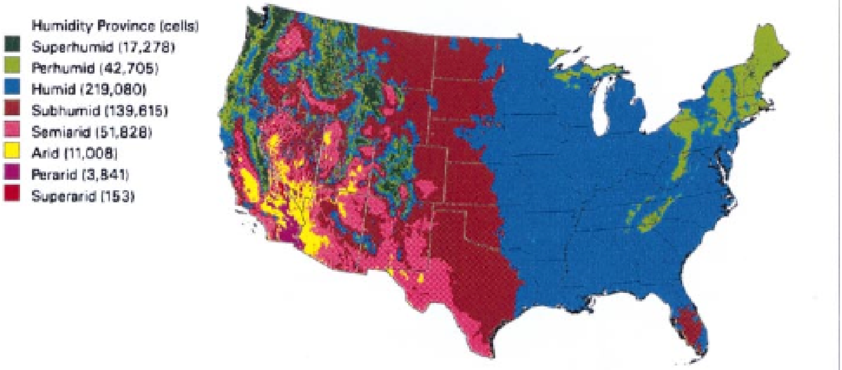
Sources of Vagueness: Ontology alignment (schema matching)

- To which degree are two concepts of two ontologies similar?



Sources of Vagueness: Lifezone mapping

- To which degree do certain areas have a specific bioclima



Holdridge life zones of USA

Sources of Vagueness: ARPAT, Air quality in the province of Lucca

I dati di domenica 21/02/2010

Sintesi dei dati rilevati dalle ore 0 alle ore 24 del giorno domenica 21/02/2010

Stazione		Tipo stazione	SO_2 µg/m³ (media su 24h)	NO_2 µg/m³ (max oraria)	CO mg/m³ (max oraria)	O_3 µg/m³ (max oraria)	PM_{10} µg/m³ (media su 24h)	Giudizio di qualità dell'aria
Lucca	P.za San Micheletto (RETE REGIONALE **)	urbana - traffico	1	75	---	---	37	Accettabile
Lucca	V.le Carducci	urbana - traffico	1	---	2,3	---	49	Accettabile
Lucca	Carignano (RETE REGIONALE **)	rurale - fondo	---	---	---	86 (h.16*)	---	Buona
Viareggio	Largo Risorgimento	urbana - traffico	---	---	1,8	---	n.d.	Buona
Viareggio	Via Maroncelli (RETE REGIONALE **)	urbana - fondo	4	97	---	61 (h.15*)	33	Accettabile
Capannori	V. di Piaggia (RETE REGIONALE **)	urbana - fondo	---	62	1,3	---	25	Accettabile
Porcari	V. Carrara (RETE REGIONALE **)	periferica - fondo	1	51	---	84 (h.15*)	24	Accettabile

Giudizio di qualità	SO_2 µg/m³ (media su 24h)	NO_2 µg/m³ (max oraria)	CO mg/m³ (max oraria)	O_3 µg/m³ (max oraria)	PM_{10} µg/m³ (media su 24h)
Buona	0-50	0-50	0-2,5	0-120	0-25
Accettabile	51-125	51-200	2,6-15	121-180	26-50
Scadente	126-250	201-400	15,1-30	181-240	51-74
Pessima	>250	>400	>30	>240	>74

Sintesi dei dati rilevati dalle ore 0 alle ore 24 del giorno domenica 14/02/2010

Stazione		Tipo stazione	SO_2 µg/m³ (media su 24h)	NO_2 µg/m³ (max oraria)	CO mg/m³ (max oraria)	O_3 µg/m³ (max oraria)	PM_{10} µg/m³ (media su 24h)	Giudizio di qualità dell'aria
Lucca	P.za San Micheletto (RETE REGIONALE **)	urbana - traffico	1	75	---	---	56	Scadente
Lucca	V.le Carducci	urbana - traffico	2	---	2	---	75	Pessima
Lucca	Carignano (RETE REGIONALE **)	rurale - fondo	---	---	---	87 (h.18*)	---	Buona
Viareggio	Largo Risorgimento	urbana - traffico	---	---	1,7	---	n.d.	Buona
Viareggio	Via Maroncelli (RETE REGIONALE **)	urbana - fondo	1	121	---	60 (h.17*)	45	Accettabile
Capannori	V. di Piaggia (RETE REGIONALE **)	urbana - fondo	---	79	2	---	59	Scadente
Porcari	V. Carrara (RETE REGIONALE **)	periferica - fondo	2	72	---	82 (h.16*)	63	Scadente

Giudizio di qualità	SO_2 µg/m³ (media su 24h)	NO_2 µg/m³ (max oraria)	CO mg/m³ (max oraria)	O_3 µg/m³ (max oraria)	PM_{10} µg/m³ (media su 24h)
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TripAdvisor: Hotel User Judgments

Uncertainty vs Vagueness: a clarification

- Initial difficulty:
  - Understand the conceptual differences between uncertainty and vagueness
- Main problem:
  - Interpreting a degree as a measure of uncertainty rather than as a measure of vagueness

Uncertain Statements

- A statement is true or false in any world/interpretation
  - We are "uncertain" about which world to consider
  - We may have e.g. a probability distribution over possible worlds
- E.g., "it will rain tomorrow"
  - We cannot exactly establish whether it will rain tomorrow or not, due to our incomplete knowledge about our world
  - We can estimate to which degree this is probable

- Consider a propositional statement (formula) ϕ
- Interpretation (world) $\mathcal{I} \in \mathcal{W}$,

$$\mathcal{I} : \mathcal{W} \to \{0, 1\}$$

- $\mathcal{I}(\phi) = 1$ means ϕ is true in $\mathcal{I}$, denoted $\mathcal{I} \models \phi$
- Each interpretation $\mathcal{I}$ depicts some concrete world
- Given n propositional letters, $|\mathcal{W}| = 2^n$
- In uncertainty theory, we do not know which interpretation $\mathcal{I}$ is the actual one

- One may construct a probability distribution over the worlds

$$Pr : \mathcal{W} \to [0, 1]$$
$$\sum_{\mathcal{I}} Pr(\mathcal{I}) = 1$$

- $Pr(\mathcal{I})$ indicates the probability that $\mathcal{I}$ is the actual world
- Probability $Pr(\phi)$ of a statement ϕ in Pr

$$Pr(\phi) = \sum_{\mathcal{I} \models \phi} Pr(\mathcal{I})$$

- $Pr(\phi)$ is the probability of the event: "ϕ is true"

Vague Statements

- A statement is true to some degree, which is taken from a truth space (usually $[0,1]$)
- The convention prescribing that a proposition is either true or false is changed towards graded propositions
- E.g., "heavy rain"
  - The compatibility of "heavy" in the phrase "heavy rain" is graded and the degree depends on the amount of rain is falling
    - The intensity of precipitation is expressed in terms of a precipitation rate R: volume flux of precipitation through a horizontal surface, i.e. $m^3/m^2s = ms^{-1}$
    - It is usually expressed in mm/h

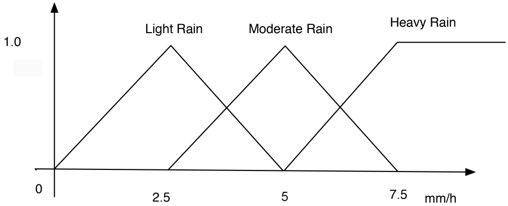
"Heavy rain" continued...E.g., in weather forecasts one may find:

- Rain intensity measured as precipitation rate R: volume flux of precipitation through a horizontal surface, i.e. $m^3/m^2h = mh^{-1}$

Rain.	Falling drops of water larger than 0.5 mm in diameter. "Rain" usually implies that the rain will fall steadily over a period of time;
Light rain.	Rain falls at the rate of 2.6 mm or less an hour;
Moderate rain.	Rain falls at the rate of 2.7 mm to 7.6 mm an hour;
Heavy rain.	Rain falls at the rate of 7.7 mm an hour or more.

- Quite harsh distinction: $$\begin{array}{lcl} R = 7.7mm/h & \rightarrow & \text{heavy rain} \\ R = 7.6mm/h & \rightarrow & \text{moderate rain} \end{array}$$
- This is clearly unsatisfactory, as quite naturally
  - The more rain is falling, the more the sentence "heavy rain" is true
  - Vice-versa, the less rain is falling the less the sentence is true

- In other words, that the sentence “heavy rain” is no longer either true or false, but is intrinsically graded
  - Even if we have complete knowledge about the current world, i.e. exact specification of the precipitation rate
- More fine grained approach:
  - Define the various types of rains as



  - Light rain, moderate rain and heavy rain are vague concepts

- Consider a propositional statement ϕ
- A propositional interpretation $\mathcal{I}$ maps ϕ to a truth degree in $[0, 1]$

$$\mathcal{I}(\phi) \in [0, 1]$$

- I.e., we are unable to establish whether a statement is entirely true or false due the occurrence of vague concept
- Vague statements are truth-functional
  - Degree of truth of a statement can be calculated from the degrees of truth of its constituents
  - Note that this is not possible for uncertain statements
- Example of truth functional interpretation of vague statements:

$$\begin{array}{rcl}\mathcal{I}(\phi \land \psi) & = & \min(\mathcal{I}(\phi), \mathcal{I}(\psi)) \\ \mathcal{I}(\phi \lor \psi) & = & \max(\mathcal{I}(\phi), \mathcal{I}(\psi)) \\ \mathcal{I}(\neg \phi) & = & 1 - \mathcal{I}(\phi)\end{array}$$

Uncertain Vague Statements

- Recap:
  - In a probabilistic setting each statement is either true or false, but there is e.g. a probability distribution telling us how probable each interpretation/sentence is

$$\mathcal{I}(\phi) \in \{0, 1\}, Pr(\mathcal{I}) \in [0, 1] \text{ and } Pr(\phi) = \sum_{\mathcal{I} \models \phi} Pr(\mathcal{I}) \quad \in [0, 1]$$

  - In vagueness theory instead, sentences are graded

$$\mathcal{I}(\phi) \in [0, 1]$$

- Are there sentences combining the two orthogonal concepts of uncertainty and vagueness?
- Yes, and we use them daily !
  - E.g. “there will be heavy rain tomorrow"
- This type of sentences are called uncertain vague sentences
- Essentially, there is
  - uncertainty about the world we will have tomorrow
  - vagueness about the various types of rain

- Consider a propositional statement ϕ
- A model for uncertain vague sentences:
  - Define probability distribution over worlds $\mathcal{I} \in \mathcal{W}$, i.e.

$$Pr(\mathcal{I}) \in [0,1], \sum_{\mathcal{I}} Pr(\mathcal{I}) = 1$$

  - Sentences are graded: each interpretation $\mathcal{I} \in W$ is truth functional and maps sentences into $[0, 1]$

$$\mathcal{I}(\phi) \in [0,1]$$

  - For a sentence ϕ, consider the expected truth of ϕ

$$ET(\phi) = \sum_{\mathcal{I}} Pr(\mathcal{I}) \cdot \mathcal{I}(\phi) .$$

- Note: if $\mathcal{I}$ is bivalent (that is, $\mathcal{I}(\phi) \in \{0,1\}$) then $ET(\phi) = Pr(\phi)$

From Fuzzy Sets to Mathematical Fuzzy Logic

Fuzzy Sets Basics

From Crisp Sets to Fuzzy Sets.

- Let X be a universal set of objects
- The power set, denoted 2^A, of a set $A \subset X$, is the set of subsets of A, i.e.,

$$2^A = \{B \mid B \subseteq A\}$$

- Often sets are defined as

$$A = \{x \mid P(x)\}$$

  - $P(x)$ is a statement "x has property P"
  - $P(x)$ is either true or false for any $x \in X$

- Examples of universe X and subsets $A, B \in 2^X$ may be

$$\begin{aligned} X &= \{x \mid x \text{ is a day}\} \\ A &= \{x \mid x \text{ is a rainy day}\} \\ B &= \{x \mid x \text{ is a day with precipitation rate } R \geq 7.5mm/h\} \end{aligned}$$

- In the above case: $B \subseteq A \subseteq X$
- The membership function of a set $A \subseteq X$:

$$\chi_A \colon X \to \{0, 1\}$$

  where $\chi_A(x) = 1$ iff $x \in A$
- Note that for sets $A, B \in 2^X$

$$A \subseteq B \text{ iff } \forall x \in X.\ \chi_A(x) \leq \chi_B(x)$$

- Complement of a set A, i.e. $\bar{A} = X \setminus A$: $\forall x \in X$:

$$\chi_{\bar{A}}(x) = 1 - \chi_A(x)$$

- Intersection and union: $\forall x \in X$

$$\begin{array}{rcl}\chi_{A \cap B}(x) & = & \min(\chi_A(x), \chi_B(x)) \\ \chi_{A \cup B}(x) & = & \max(\chi_A(x), \chi_B(x))\end{array}$$

- Cartesian product of two sets $A, B \in 2^X$

$$A \times B = \{\langle a, b\rangle \mid a \in A, b \in B\}$$

- A relation $R \subseteq X \times X$
  - is reflexive if for all $x \in X$

$$\chi_R(x, x) = 1$$

  - is symmetric if for all $x, y \in X$

$$\chi_R(x, y) = \chi_R(y, x)$$

  - Inverse of R, $\chi_{R^{-1}} : X \times X \to \{0, 1\}$: $\forall x, y \in X$:

$$\chi_{R^{-1}}(y, x) = \chi_R(x, y)$$

- Fuzzy set A: $\chi_A \colon X \to [0,1]$, or simply

$$A \colon X \to [0,1]$$

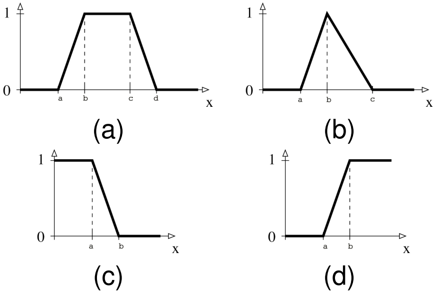
- Fuzzy power set over X, is denoted $\tilde{2}^X$, i.e. the set of all fuzzy sets over X
- Example: the fuzzy set

$$C = \{x \mid x \text{ is a day with heavy precipitation rate } R\}$$

is defined via the membership function

$$\chi_C(x) = \begin{cases} 1 & \text{if } R \geq 7.5 \\ (x-5)/2.5 & \text{if } R \in [5, 7.5) \\ 0 & \text{otherwise} \end{cases}$$

- Fuzzy membership functions may depend on the context and may be subjective
- Shape may be quite different
- Usually, it is sufficient to consider functions

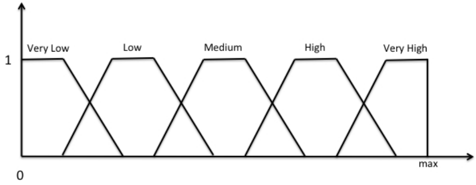


(a) Trapezoidal *trz*(a, b, c, d); (b) Triangular *tri*(a, b, c); (c) left-shoulder *ls*(a, b); (d) right-shoulder *rs*(a, b)

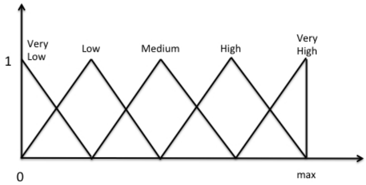
Fuzzy Sets Construction

- The usefulness of fuzzy sets depends critically on appropriate membership functions
- Methods for fuzzy membership functions construction is largely addressed in literature

- Easy and typically satisfactory method (numerical domain)
  - uniform partitioning into 5 fuzzy sets

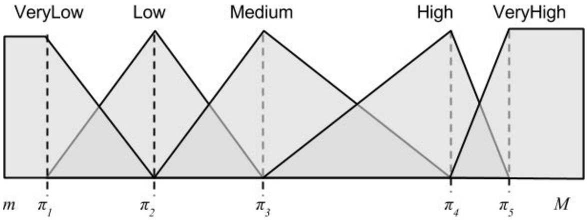


Fuzzy sets construction using trapezoidal functions



Fuzzy sets construction using triangular functions

- Another popular method is based on clustering
- Use Fuzzy C-Means to cluster data into 5 clusters
  - Fuzzy C-Means extends K-Means to accommodates graded membership
- From the clusters $c_1, \ldots, c_5$ take the centroids $\pi_1, \ldots, \pi_5$
- Build the fuzzy sets from the centroids



Fuzzy sets construction using clustering

Norm-Based Fuzzy Set Operations

- Standard fuzzy set operations are not the only ones
- Most notable ones are triangular norms
  - t-norm $\otimes$ for set intersection
  - t-conorm $\oplus$ (also called s-norm) for set union
  - negation $\ominus$ for set complementation
  - implication $\Rightarrow$
    - set inclusion $A \sqsubseteq B$ is defined as

$$\inf_{x \in X} A(x) \Rightarrow B(x)$$

    - $\Rightarrow$ is often defined from $\otimes$ as *r-implication*

$$a \Rightarrow b = \sup \{c \mid a \otimes c \leq b\} \ .$$

- These functions satisfy some properties that one expects to hold

Properties for t-norms and s-norms

Axiom Name	T-norm	S-norm
Taututology/Contradiction	$a \otimes 0 = 0$	$a \oplus 1 = 1$
Identity	$a \otimes 1 = a$	$a \oplus 0 = a$
Commutativity	$a \otimes b = b \otimes a$	$a \oplus b = b \oplus a$
Associativity	$(a \otimes b) \otimes c = a \otimes (b \otimes c)$	$(a \oplus b) \oplus c = a \oplus (b \oplus c)$
Monotonicity	if $b \leq c$, then $a \otimes b \leq a \otimes c$	if $b \leq c$, then $a \oplus b \leq a \oplus c$

Properties for implication and negation functions

Axiom Name	Implication Function	Negation Function
Tautology / Contradiction	$0 \Rightarrow b = 1,\ a \Rightarrow 1 = 1,\ 1 \Rightarrow 0 = 0$	$\ominus 0 = 1,\ \ominus 1 = 0$
Antitonicity	if $a \leq b$, then $a \Rightarrow c \geq b \Rightarrow c$	if $a \leq b$, then $\ominus a \geq \ominus b$
Monotonicity	if $b \leq c$, then $a \Rightarrow b \leq a \Rightarrow c$	

- By commutativity, $\otimes$ and $\oplus$ are monotone also in the first argument
- $\otimes$ is indempotent if $a \otimes a = a$, for all $a \in [0, 1]$
- Megation function $\ominus$ is involutive iff $\ominus \ominus a = a$, for all $a \in [0, 1]$.
- Salient negation functions are:
  Standard or Łukasiewicz negation: $\ominus_l a = 1 - a$;
  Gödel negation: $\ominus_g a$ is 1 if $a = 0$, else is 0.
- Łukasiewicz negation is involutive, Gödel negation is not.

- Salient t-norm functions are:

  Gödel t-norm: $a \otimes_g b = \min(a, b)$;

  Bounded difference or Łukasiewicz t-norm:

  $$a \otimes_l b = \max(0, a + b - 1);$$

  Algebraic product or product t-norm: $a \otimes_p b = a \cdot b$;

  Drastic product: $a \otimes_d b =$

  $$\begin{cases} 0 & \text{when } (a, b) \in [0, 1[\times[0, 1[\\ \min(a, b) & \text{otherwise} \end{cases}$$

- Salient s-norm functions are:

  Gödel s-norm: $a \oplus_g b = \max(a, b)$;

  Bounded sum or Łukasiewicz s-norm:

  $$a \oplus_l b = \min(1, a + b);$$

  Algebraic sum or product s-norm: $a \oplus_p b = a + b - ab$;

  Drastic sum: $a \oplus_d b =$

  $$\begin{cases} 1 & \text{when } (a, b) \in]0,1] \times]0,1] \\ \max(a, b) & \text{otherwise} \end{cases}$$

Salient properties of norms:

- Ordering among t-norms ($\otimes$ is any t-norm):

$$\otimes_d \leq \otimes \leq \otimes_g$$
$$\otimes_d \leq \otimes_l \leq \otimes_p \leq \otimes_g \ .$$

- The only idempotent t-norm is $\otimes_g$.
- The only t-norm satisfying $a \otimes a = 0$ for all $a \in [0, 1[$ is $\otimes_d$.
- Ordering among s-norms ($\oplus$ is any s-norm):

$$\oplus_g \leq \oplus \leq \oplus_d$$
$$\oplus_g \leq \oplus_p \leq \oplus_l \leq \oplus_d \ .$$

- The only idempotent s-norm is $\oplus_g$.
- The only s-norm satisfying $a \oplus a = 1$ for all $a \in]0, 1]$ is $\oplus_d$.
- The dual s-norm of $\otimes$ is defined as

$$a \oplus b = 1 - (1 - a) \otimes (1 - b) .$$

- Kleene-Dienes implication: $x \Rightarrow y = \max(1 - x, y)$ is called
- Fuzzy modus ponens: let $a \geq n$ and $a \Rightarrow b \geq m$
  - Under Kleene-Dienes implication, we infer that if $n > 1 - m$ then $b \geq m$
  - Under r-implication relative to a t-norm $\otimes$, we infer that $b \geq n \otimes m$
- composition of two fuzzy relations $R_1 : X \times X \to [0, 1]$ and $R_2 : X \times X \to [0, 1]$: for all $x, z \in X$
  - $(R_1 \circ R_2)(x, z) = \sup_{y \in X} R_1(x, y) \otimes R_2(y, z)$
- A fuzzy relation R is transitive iff for all $x, z \in X$ $R(x, z) \geq (R \circ R)(x, z)$

Łukasiewicz, Gödel, Product logic and Standard Fuzzy logic

- One distinguishes three different sets of fuzzy set operations (called fuzzy logics)
  - Łukasiewicz, Gödel, and Product logic
  - Standard Fuzzy Logic (SFL) is a sublogic of Łukasiewicz
    - $\min(a, b) = a \otimes_l (a \Rightarrow_l b)$, $\max(a, b) = 1 - \min(1 - a, 1 - b)$

	Łukasiewicz Logic	Gödel Logic	Product Logic	SFL
$a \otimes b$	$\max(a + b - 1, 0)$	$\min(a, b)$	$a \cdot b$	$\min(a, b)$
$a \oplus b$	$\min(a + b, 1)$	$\max(a, b)$	$a + b - a \cdot b$	$\max(a, b)$
$a \Rightarrow b$	$\min(1 - a + b, 1)$	$\begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$	$\min(1, b/a)$	$\max(1 - a, b)$
$\ominus a$	$1 - a$	$\begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$	$1 - a$

- Mostert–Shields theorem: any continuous t-norm can be obtained as an ordinal sum of these three

Some additional properties

Property	Łukasiewicz Logic	Gödel Logic	Product Logic	SFL
$x \otimes \ominus x = 0$	•			
$x \oplus \ominus x = 1$	•			
$x \otimes x = x$		•		•
$x \oplus x = x$		•		•
$\ominus \ominus x = x$	•			•
$x \Rightarrow y = \ominus x \oplus y$	•			•
$\ominus (x \Rightarrow y) = x \otimes \ominus y$	•			•
$\ominus (x \otimes y) = \ominus x \oplus \ominus y$	•	•	•	•
$\ominus (x \oplus y) = \ominus x \otimes \ominus y$	•	•	•	•

- Note: If all conditions in the upper part of a column have to be satisfied then we collapse to classical two-valued logic

Fuzzy Modifiers

- Fuzzy modifiers: interesting feature of fuzzy set theory
- A fuzzy modifier apply to fuzzy sets to change their membership function
  - Examples: very, more_or_less, and slightly
- A fuzzy modifier m represents a function

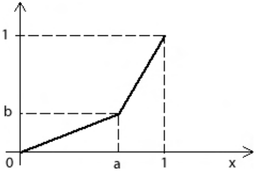
$$f_m \colon [0,1] \to [0,1]$$

Example: $f_{\texttt{very}}(x) = x^2$, $f_{\texttt{more\_or\_less}}(x) = tri(0, x, 1)$, $f_{\texttt{slightly}}(x) = \sqrt{x}$

- Modelling the fuzzy set of very heavy rain:

$$\begin{aligned}\chi_{\texttt{very heavy rain}}(x) &= f_{\texttt{very}}(\chi_{\texttt{heavyrain}}(x)) \\ &= (\chi_{\texttt{heavyrain}}(x))^2 \\ &= (rs(5, 7.5)(x))^2\end{aligned}$$

- A typical shape of modifiers: linear modifiers $lm(a, b)$



- Note: linear modifiers require one parameter c only

$$lm(a, b) = lm(c)$$

where $a = c/(c + 1)$, $b = 1/(c + 1)$

Mathematical Fuzzy Logics Basics

- OWL 2 is grounded on Mathematical Logic
- Fuzzy OWL 2 is grounded on Mathematical Fuzzy Logic
- A statement is no longer either true or false, but is graded
- Truth space: set of truth values L with some structure
- Given a statement ϕ
  - Fuzzy Interpretation: a function $\mathcal{I}$ mapping ϕ into L, i.e.

$$\mathcal{I}(\varphi) \in L$$

  - Usually

$$\begin{aligned} L &= [0,1] \\ L_n &= \{0, \frac{1}{n}, \ldots, \frac{n-2}{n-1}, \ldots, 1\} \quad (n \geq 1) \end{aligned}$$

- Fuzzy statement: for $r \in [0, 1]$

$$\langle \phi, r \rangle$$

The degree of truth of ϕ is equal or greater than r

- Examples:
  - Fuzzy FOL: $\langle \mathit{RainyDay}(d), 0.75 \rangle$
  - Fuzzy LPs: $\langle \mathit{RainyDay}(d) \leftarrow, 0.75 \rangle$
  - Fuzzy RDFS: $\langle \langle d, \mathit{type}, \mathit{RainyDay} \rangle, 0.75 \rangle$
  - Fuzzy DLs: $\langle d{:}\mathit{RainyDay}, 0.75 \rangle$

- Fuzzy interpretation $\mathcal{I}$:
  - Maps each basic statement p_i into $[0,1]$
  - Extended inductively to all statements

$$
\begin{array}{lcl}
\mathcal{I}(\phi \wedge \psi) & = & \mathcal{I}(\phi) \otimes \mathcal{I}(\psi) \\
\mathcal{I}(\phi \vee \psi) & = & \mathcal{I}(\phi) \oplus \mathcal{I}(\psi) \\
\mathcal{I}(\phi \to \psi) & = & \mathcal{I}(\phi) \Rightarrow \mathcal{I}(\psi) \\
\mathcal{I}(\phi \leftrightarrow \psi) & = & \mathcal{I}(\phi \to \psi) \otimes \mathcal{I}(\psi \to \phi) \\
\mathcal{I}(\neg \phi) & = & \ominus \mathcal{I}(\phi) \\
\mathcal{I}(\exists x.\phi) & = & \sup_{a \in \Delta^{\mathcal{I}}} \mathcal{I}^a_x(\phi) \\
\mathcal{I}(\forall x.\phi) & = & \inf_{a \in \Delta^{\mathcal{I}}} \mathcal{I}^a_x(\phi) \ ,
\end{array}
$$

where
  - $\Delta^{\mathcal{I}}$ is the domain of $\mathcal{I}$
  - $\otimes$, $\oplus$, $\Rightarrow$, and $\ominus$ are the t-norms, t-conorms, implication functions, a negation functions
  - The function $\mathcal{I}^a_x$ is as $\mathcal{I}$ except that x is interpreted as a

Example

- In Lukasiewicz logic:

$$\varphi = \mathit{Cold} \wedge \mathit{Cloudy}$$

$\mathcal{I}$	*Cold*	*Cloudy*	$\mathcal{I}(\varphi)$
$\mathcal{I}_1$	0	0.1	$\max(0, 0 + 0.1 - 1) = 0.0$
$\mathcal{I}_2$	0.3	0.4	$\max(0, 0.3 + 0.4 - 1) = 0.0$
$\mathcal{I}_3$	0.7	0.8	$\max(0, 0.7 + 0.9 - 1) = 0.6$
$\mathcal{I}_4$	1	1	$\max(0, 1 + 1 - 1) = 1.0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

- Note: given *m* propositional letters
  - Fuzzy interpretations over $L = [0, 1]$ are not recursively enumerable
  - There are n^m fuzzy interpretations over L_n

- One may also consider the following abbreviations:

$$
\begin{aligned}
\phi \wedge_g \psi &\stackrel{\text{def}}{=} \phi \wedge (\phi \to \psi) \\
\phi \vee_g \psi &\stackrel{\text{def}}{=} (\phi \to \psi) \to \phi) \wedge_g (\psi \to \phi) \to \psi) \\
\neg_{\otimes}\phi &\stackrel{\text{def}}{=} \phi \to 0 \\
\langle \phi \leq r \rangle &\stackrel{\text{def}}{=} \langle \neg_l \phi, 1 - r \rangle
\end{aligned}
$$

- In case $\Rightarrow$ is the r-implication based on $\otimes$, then
  - $\wedge_g$ is Gödel t-norm
  - $\vee_g$ is Gödel s-norm
  - $\neg_{\otimes}$ is interpreted as the negation function related to $\otimes$

- $\mathcal{I}$ satisfies $\langle \phi, r \rangle$, or $\mathcal{I}$ is a model of $\langle \phi, r \rangle$

$$\mathcal{I} \models \langle \phi, r \rangle \text{ iff } \mathcal{I}(\phi) \geq r$$

- $\mathcal{I}$ is a model of ϕ if $\mathcal{I}(\phi) = 1$
- Fuzzy knowledge base $\mathcal{K}$: finite set of fuzzy statements
- $\mathcal{I}$ satisfies (is a model of) $\mathcal{K}$: $\mathcal{I} \models \mathcal{K}$ iff it satisfies each element in it
- Best entailment degree of ϕ w.r.t. $\mathcal{K}$:

$$bed(\mathcal{K}, \phi) = \sup \{r \mid \mathcal{K} \models \langle \phi, r \rangle\}$$

- Best satisfiability degree of ϕ w.r.t. $\mathcal{K}$:

$$bsd(\mathcal{K}, \phi) = \sup_{\mathcal{I}} \{\mathcal{I}(\phi) \mid \mathcal{I} \models \mathcal{K}\}$$

Proposition (Fuzzy Modus Ponens)

For r-implication $\to$, for $r, s \in [0,1]$:

$$\langle \phi, r \rangle, \langle \phi \to \psi, s \rangle \models \langle \psi, r \otimes s \rangle$$

Proposition

Salient equivalences:

$$\begin{array}{rcl} \neg\neg\phi & \equiv & \phi \;\; (Ł, SFL) \\ \phi \wedge \phi & \equiv & \phi \;\; (G, SFL) \\ \neg(\phi \wedge \neg\phi) & \equiv & 1 \;\; (Ł, G, \Pi) \\ \phi \vee \neg\phi & \equiv & 1 \;\; (Ł) \\ \forall x.\phi & \equiv & \neg\exists x.\neg\phi \;\; (Ł, SFL) \end{array}$$

Proposition

Salient equivalences:

$$
\begin{aligned}
Ł + G &\equiv \textit{Boolean Logic} \\
Ł + \Pi &\equiv \textit{Boolean Logic} \\
G + \Pi &\equiv \textit{Boolean Logic}
\end{aligned}
$$

Proposition (BED)

$bed(\mathcal{K}, \phi) = \min x.$ *such that* $\mathcal{K} \cup \{\langle \varphi \leq x \rangle\}$ *satisfiable.*

Proposition (BSD)

$bsd(\mathcal{K}, \phi) = \max x.$ *such that* $\mathcal{K} \cup \{\langle \varphi, x \rangle\}$ *satisfiable.*

On Witnessed Models

- Witnessed interpretation $\mathcal{I}$:

$$\mathcal{I}(\exists x.\phi) = \mathcal{I}_x^a(\phi), \text{ for some } a \in \Delta^{\mathcal{I}} \quad (1)$$

$$\mathcal{I}(\forall x.\phi) = \mathcal{I}_x^a(\phi), \text{ for some } a \in \Delta^{\mathcal{I}} \quad (2)$$

- The supremum (resp. infimum) are attained at some point
- Classical interpretations are witnessed
- Fuzzy interpretations may not be witnessed
- E.g., $\mathcal{I}$ is not witnessed as Eq. (1) not satisfied:

$$\begin{aligned}
\Delta^{\mathcal{I}} &= \mathbb{N} \\
\mathcal{I}_x^n(A(x)) &= 1 - 1/n < 1, \text{ for all } n \\
\mathcal{I}(\exists x.A(x)) &= \sup_n \mathcal{I}_x^n(A(x)) \\
&= \sup_n 1 - 1/n = 1
\end{aligned}$$

Proposition (Witnessed model property)

In Łukasiewicz logic and SFL over $L = [0, 1]$*, or for all cases in which the truth space L is finite, a fuzzy KB has a witnessed fuzzy model iff it has a fuzzy model.*

- Not true for Gödel and product logic over $L = [0, 1]$
  - $\neg\forall x\, p(x) \land \neg\exists x\, \neg p(x)$ has no classical model
  - In Gödel logic it has no finite model, but has an infinite model: for integer $n \geq 1$, let $\mathcal{I}$ such that $\mathcal{I}(p(n)) = 1/n$

$$
\begin{aligned}
\mathcal{I}(\forall x\, p(x)) &= \inf_n 1/n = 0 \\
\mathcal{I}(\exists x\, \neg p(x)) &= \sup_n \neg 1/n = \sup 0 = 0
\end{aligned}
$$

- IMHO: non-witnessed models make little sense in KR
- We will always assume that interpretations are witnessed

Fuzzy Propositional Logic: Reasoning

- We need to distinguish if truth space is $L = [0, 1]$ or $L_n = \{0, \frac{1}{n}, \ldots, \frac{n-2}{n-1}, \ldots, 1\}$
- Case L_n easier: given m propositional letters, there are m^n possible interpretations
- We may use
  - Operational Research
  - Analytic Tableaux, Non-Deterministic Analytic Tableaux
  - Reduction into Classical Propositional Logic

Operational Research: Case Łukasiewicz Logic & SFL

- Basic idea: translate formulae into equational constraints about truth degrees
- For a formula ϕ consider a variable x_ϕ
  - Intuition: x_ϕ will hold the degree of truth of statement ϕ
  - Example: constraints under Łukasiewicz for $\langle \neg\phi, 0.6 \rangle$

$$\begin{aligned} x_{\neg\phi} &\in [0,1] \\ x_\phi &\in [0,1] \\ x_{\neg\phi} &= 1 - x_\phi \end{aligned}$$

- We may use Mixed Integer Linear Programming for the encodings of constraints

  For Łukasiewicz:

  - $x_1 \otimes_l x_2 = z$
    $\mapsto \{x_1 + x_2 - 1 \leq z, x_1 + x_2 - 1 \geq z - y, z \leq 1 - y, y \in \{0, 1\}\}$,
    where y is a new variable.
  - $x_1 \oplus_l x_2 = z \mapsto \{x_1 + x_2 \leq z + y, y \leq z, x_1 + x_2 \geq z, y \in \{0, 1\}\}$,
    where y is a new variable.
  - $x_1 \Rightarrow_l x_2 = z \mapsto \{(1 - x_1) \oplus_l x_2 = z\}$.

  For SFL:

  - $x_1 \otimes_g x_2 = z$
    $\mapsto \{z \leq x_1, z \leq x_2, x_1 \leq z + y, x_2 \leq z + (1 - y), y \in \{0, 1\}\}$,
    where y is a new variable.
  - $x_1 \oplus_g x_2 = z$
    $\mapsto \{z \geq x_1, z \geq x_2, x_1 + y \geq z, x_2 + (1 - y) \geq z, y \in \{0, 1\}\}$,
    where y is a new variable.
  - $x_1 \Rightarrow_{kd} x_2 = z \mapsto (1 - x_1) \oplus_g x_2 = z$.

- Negation Normal Form, $\mathit{nnf}(\phi)$

$$
\begin{array}{rcl}
\neg\perp & = & \top \\
\neg\top & = & \perp \\
\neg\neg\phi & \mapsto & \phi \\
\neg(\phi \wedge \psi) & \mapsto & \neg\phi \vee \neg\psi \\
\neg(\phi \vee \psi) & \mapsto & \neg\phi \wedge \neg\psi \\
\neg(\phi \rightarrow \psi) & \mapsto & \phi \wedge \neg\psi \ .
\end{array}
$$

1. Transform $\mathcal{K}$ into NNF
2. Initialize the fuzzy theory $\mathcal{T}_\mathcal{K}$ and the initial set of constraints $\mathcal{C}_\mathcal{K}$ by

$$\begin{array}{lcl} \mathcal{T}_\mathcal{K} & = & \{\phi \mid \langle \phi, n \rangle \in \mathcal{K}\} \\ \mathcal{C}_\mathcal{K} & = & \{x_\psi \geq n \mid \langle \phi, n \rangle \in \mathcal{K}\} \end{array}$$

3. Apply the following inference rules until no more rules can be applied

(var). For variable x_ϕ occurring in $\mathcal{C}_\mathcal{K}$ add $x_\phi \in [0,1]$ to $\mathcal{C}_\mathcal{K}$
($\bar{var}$). For variable $x_{\neg\phi}$ occurring in $\mathcal{C}_\mathcal{K}$ add $x_\phi = 1 - x_{\neg\phi}$ to $\mathcal{C}_\mathcal{K}$
($\bot$). If $\bot \in \mathcal{T}_\mathcal{K}$ then $\mathcal{C}_\mathcal{K} := \mathcal{C}_\mathcal{K} \cup \{x_\bot = 0\}$
($\top$). If $\top \in \mathcal{T}_\mathcal{K}$ then $\mathcal{C}_\mathcal{K} := \mathcal{C}_\mathcal{K} \cup \{x_\top = 1\}$

($\wedge$). If $\phi \wedge \psi \in \mathcal{T}_\mathcal{K}$, then
1. add ϕ and ψ to $\mathcal{T}_\mathcal{K}$
2. $\mathcal{C}_\mathcal{K} := \mathcal{C}_\mathcal{K} \cup \{x_\phi \otimes x_\psi = x_{\phi\wedge\psi}\}$

($\vee$). If $\phi \vee \psi \in \mathcal{T}_\mathcal{K}$, then
1. add ϕ and ψ to $\mathcal{T}_\mathcal{K}$
2. $\mathcal{C}_\mathcal{K} := \mathcal{C}_\mathcal{K} \cup \{x_\phi \oplus x_\psi = x_{\phi\wedge\psi}\}$

($\rightarrow$). If $\phi \rightarrow \psi \in \mathcal{T}_\mathcal{K}$, then
1. add $nnf(\neg\phi)$ and ψ to $\mathcal{T}_\mathcal{K}$
2. $\mathcal{C}_\mathcal{K} := \mathcal{C}_\mathcal{K} \cup \{(1 - x_{nnf(\neg\phi)}) \Rightarrow x_\psi = x_{\phi\rightarrow\psi}\}$

sat$(\mathcal{K})$: $\mathcal{K}$ is satisfiable iff the final set of constraints $\mathcal{C}_{\mathcal{K}}$ has a solution

bed$(\mathcal{K}, \phi)$:
- Add $\neg\phi$ to $\mathcal{T}_{\mathcal{K}}$
- Add $x_{\neg\phi} \geq 1 - x, x \in [0, 1]$ to $\mathcal{C}_{\mathcal{K}}$, x new
- Compute final set of constraints $\mathcal{C}_{\mathcal{K}}$
- Then, solve the optimisation problem

$$bed(\mathcal{K}, \phi) = \min x.\ \text{such that } \mathcal{C}_{\mathcal{K}} \text{ has a solution}$$

bsd$(\mathcal{K}, \phi)$:
- Add ϕ to $\mathcal{T}_{\mathcal{K}}$
- Add $x_{\phi} \geq x, x \in [0, 1]$ to $\mathcal{C}_{\mathcal{K}}$, x new
- Compute final set of constraints $\mathcal{C}_{\mathcal{K}}$
- Then, solve the optimisation problem

$$bsd(\mathcal{K}, \phi) = \max x.\ \text{such that } \mathcal{C}_{\mathcal{K}} \text{ has a solution}$$

Analytical Fuzzy Tableau: Case SFL

- Main property the method is based on:
  - if $\mathcal{I}$ is model of $\langle \phi \wedge \psi, n \rangle$ then $\mathcal{I}$ is a model of both $\langle \phi, n \rangle$ and $\langle \psi, n \rangle$;
  - if $\mathcal{I}$ is model of $\langle \phi \vee \psi, n \rangle$ then $\mathcal{I}$ is a model of either $\langle \phi, n \rangle$ or $\langle \psi, n \rangle$.
  - $\mathcal{I}$ cannot be a model of both $\langle p, n \rangle$ and $\langle \neg p, m \rangle$ if $n > 1 - m$.
- A clash is either
  - a fuzzy statement $\langle \perp, n \rangle$ with $n > 0$; or
  - a pair of fuzzy statements $\langle p, n \rangle$ and $\langle \neg p, m \rangle$ with $n > 1 - m$
- Clash-free: does not contain a clash

1. Transform $\mathcal{K}$ into NNF
2. Initialize the completion $\mathcal{S}_{\mathcal{K}} = \mathcal{K}$
3. Apply the following inference rules to $\mathcal{S}_{\mathcal{K}}$ until no more rules can be applied
4. We call a set of fuzzy statements $\mathcal{S}_{\mathcal{K}}$ complete iff none of the rules below can be applied to $\mathcal{S}_{\mathcal{K}}$
5. Note that rule $(\vee)$ is non-deterministic

$(\wedge)$. If $\langle \phi \wedge \psi, n \rangle \in \mathcal{S}_{\mathcal{K}}$ and $\{\langle \phi, n \rangle, \langle \psi, n \rangle\} \not\subseteq \mathcal{S}_{\mathcal{K}}$, then add both $\langle \phi, n \rangle$ and $\langle \psi, n \rangle$ to $\mathcal{S}_{\mathcal{K}}$

$(\vee)$. If $\langle \phi \vee \psi, n \rangle \in \mathcal{S}_{\mathcal{K}}$ and $\{\langle \phi, n \rangle, \langle \psi, n \rangle\} \cap \mathcal{S}_{\mathcal{K}} = \emptyset$, then add either $\langle \phi, n \rangle$ *or* $\langle \psi, n \rangle$ to $\mathcal{S}_{\mathcal{K}}$

$(\rightarrow)$. If $\langle \phi \rightarrow \psi, n \rangle \in \mathcal{S}_{\mathcal{K}}$ and $\langle nnf(\neg \phi) \vee \psi, n \rangle \notin \mathcal{S}_{\mathcal{K}}$, then add $\langle nnf(\neg \phi) \vee \psi, n \rangle$ to $\mathcal{S}_{\mathcal{K}}$

sat$(\mathcal{K})$: $\mathcal{K}$ is satisfiable iff we find a complete and clash-free completion $\mathcal{S}_{\mathcal{K}}$ of $\mathcal{K}$

- For BED and BSD we need some more work
- Given $\mathcal{K}$, define

$$
\begin{array}{rcl}
N^{\mathcal{K}} & = & \{0, 0.5, 1\} \ \cup \ \{n \mid \langle \phi, n \rangle \in \mathcal{K}\} \\
\bar{N}^{\mathcal{K}} & = & N^{\mathcal{K}} \cup \{1 - n \mid n \in N^{\mathcal{K}}\} \\
\epsilon & = & \min\{d/2 \mid n, m \in \bar{N}^{\mathcal{K}}, n \neq m, d = |n - m|\}
\end{array}
$$

Proposition

Under SFL, given $\mathcal{K}$, then for $n > 0$

$$\mathcal{K} \models \langle \phi, n \rangle \text{ iff } \mathcal{K} \cup \{\langle \neg\phi, 1 - n + \epsilon \rangle\} \text{ is not satisfiable .}$$

Moreover, $\mathcal{K}$ is satisfiable iff it has a model over $\bar{N}^{\mathcal{K}}$.

bed$(\mathcal{K}, \phi)$: Find greatest $n \in \bar{N}^{\mathcal{K}}$ such that $\mathcal{K} \models \langle \phi, n \rangle$

bsd$(\mathcal{K}, \phi)$: Find greatest $n \in \bar{N}^{\mathcal{K}}$ such that $\mathcal{K} \cup \{\langle \phi, n \rangle\}$ satisfiable

Non Deterministic Analytic Fuzzy Tableau

- Works for finitely-valued fuzzy propositional logic over L_n
- Works also for SFL (as in place of $[0, 1]$, we may use $\bar{N}^{\mathcal{K}}$)
- Basic idea is as for fuzzy tableau, but now we guess the truth degrees

  ($\wedge$). If $\langle \phi \wedge \psi, n \rangle \in \mathcal{S}_{\mathcal{K}}$, $n_1, n_2 \in L_n$ such that $n_1 \otimes n_2 = n$ and $\{\langle \phi, n_1 \rangle, \langle \psi, n_2 \rangle\} \not\subseteq \mathcal{S}_{\mathcal{K}}$, then add both $\langle \phi, n_1 \rangle$ and $\langle \psi, n_2 \rangle$ to $\mathcal{S}_{\mathcal{K}}$

  ($\vee$). If $\langle \phi \vee \psi, n \rangle \in \mathcal{S}_{\mathcal{K}}$, $n_1, n_2 \in L_n$ such that $n_1 \oplus n_2 = n$ and $\{\langle \phi, n_1 \rangle, \langle \psi, n_2 \rangle\} \not\subseteq \mathcal{S}_{\mathcal{K}}$, then add both $\langle \phi, n_1 \rangle$ and $\langle \psi, n_2 \rangle$ to $\mathcal{S}_{\mathcal{K}}$

  ($\rightarrow$). If $\langle \phi \rightarrow \psi, n \rangle \in \mathcal{S}_{\mathcal{K}}$, $n_1, n_2 \in L_n$ such that $n_1 \Rightarrow n_2 = n$ and $\{\langle \phi, n_1 \rangle, \langle \psi, n_2 \rangle\} \not\subseteq \mathcal{S}_{\mathcal{K}}$, then add both $\langle \phi, n_1 \rangle$ and $\langle \psi, n_2 \rangle$ to $\mathcal{S}_{\mathcal{K}}$

- A clash is either
  - a fuzzy statement $\langle \bot, n \rangle$ with $n > 0$; or
  - a pair of fuzzy statements $\langle p, n \rangle$ and $\langle \neg p, m \rangle$ such that

$$x_p \geq n, \quad \ominus x_p \geq m, x_p \in L_n$$

  has no solution

Reduction to Classical Propositional Logic: Case SFL over [0, 1]

- Given $\mathcal{K}$, we know that we can use

$$L_n = \bar{N}^{\mathcal{K}} = \{\gamma_1, \ldots, \gamma_n\}$$

  with $\gamma_i < \gamma_{i+1}, 1 \leq i \leq n-1$
- Basic idea: use atom $A_{\geq r}$ to represent

  The truth degree of A has to be equal or greater than r

- Similarly for $A_{>r}$, $A_{\leq r}$ and $A_{<r}$

- To start with, build $Crisp_{L_n}$
  - For all atoms A, for all $1 \leq i \leq n-1, 2 \leq j \leq n-1$

$$
\begin{array}{l}
A_{\geq \gamma_{i+1}} \rightarrow A_{>\gamma_i} \\
A_{>\gamma_j} \rightarrow A_{\geq \gamma_j}
\end{array}
$$

- Build $Crisp_{\mathcal{K}}$:

$$
\begin{array}{rcl}
Crisp_{\mathcal{K}} & = & \{\rho(\phi, n) \mid \langle \phi, n \rangle \in \mathcal{K}\} \cup \\
& & Crisp_{L_n} \ ,
\end{array}
$$

x	y	$\rho(x, y)$
$\top$	c	$\top$
$\bot$	0	$\top$
$\bot$	c	$\bot$ if $c > 0$
A	c	$A_{\geq c}$
$\neg A$	c	$\neg A_{>1-c}$
$\phi \wedge \psi$	c	$\rho(\phi, c) \wedge \rho(\psi, c)$
$\phi \vee \psi$	c	$\rho(\phi, c) \vee \rho(\psi, c)$

Proposition

Given $\mathcal{K}$ under SFL over L_n, then $\mathcal{K} \models \langle \phi, c \rangle$ iff $\mathcal{K} \cup \{\langle \neg\phi, 1 - c^- \rangle\}$ is not satisfiable, where c^- is the next smaller value than c in L_n

$sat(\mathcal{K})$: $\mathcal{K}$ is satisfiable iff $Crisp_{\mathcal{K}}$ satisfiable

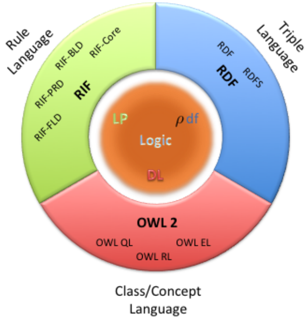
$bed(\mathcal{K}, \phi)$: Find greatest $c \in L_n$ such that $\mathcal{K} \models \langle \phi, c \rangle$

$bsd(\mathcal{K}, \phi)$: Find greatest $c \in L_n$ such that $\mathcal{K} \cup \{\langle \phi, c \rangle\}$ satisfiable

Fuzzy Description Logics and OWL 2

The Semantic Web Family of Languages

- Wide variety of languages
  - RDFS: *Triple language*, -*Resource Description Framework*
    - The logical counterpart is ρdf
  - RIF: *Rule language*, -*Rule Interchange Format*,
    - Relate to the *Logic Programming* (LP) paradigm
  - OWL 2: *Conceptual language*, -*Ontology Web Language*
    - Relate to Description Logics (DLs)



OWL 2 Profiles

OWL 2 EL

- Useful for large size of properties and/or classes
- Basic reasoning problems solved inpolynomial time
- The EL acronym refers to the $\mathcal{EL}$ family of DLs

OWL 2 QL

- Useful for very large volumes of instance data
- Conjunctive query answering via via query rewriting and SQL
- OWL 2 QL relates to the DL family *DL-Lite*

OWL 2 RL

- Useful for scalable reasoning without sacrificing too much expressive power
- OWL 2 RL maps to Datalog
- Computational complexity: same as for Datalog, polynomial in size of the data, EXPTIME w.r.t. size of knowledge base

Description Logics (DLs)

- Concept/Class: names are equivalent to unary predicates
  - In general, concepts equiv to formulae with one free variable
- Role or attribute: names are equivalent to binary predicates
  - In general, roles equiv to formulae with two free variables
- Taxonomy: Concept and role hierarchies can be expressed
- Individual: names are equivalent to constants
- Operators: restricted so that
  - Language is decidable and, if possible, of low complexity
  - No need for explicit use of variables
    - Restricted form of $\exists$ and $\forall$
  - Features such as counting can be succinctly expressed

- Basic ingredients: descriptions of classes, properties, and their instances, such as
  - $a{:}C$, meaning that individual a is an instance of concept/class C

$$\mathrm{a{:}Person} \sqcap \forall \mathrm{hasChild.Femal}$$

  - $(a, b){:}R$, meaning that the pair of individuals $\langle a, b\rangle$ is an instance of the property/role R

$$\mathrm{(tom, mary){:}hasChild}$$

  - $C \sqsubseteq D$, meaning that the class C is a subclass of class D

$$\mathrm{Person} \sqsubseteq \forall \mathrm{hasChild.Person}$$

The DL Family

- A given DL is defined by set of concept and role forming operators
- Basic language: $\mathcal{ALC}$ ($\mathcal{A}$ttributive $\mathcal{L}$anguage with $\mathcal{C}$omplement)

Syntax				Semantics	Example
C, D	$\rightarrow$	$\top$	|	$\top(x)$	
		$\bot$	|	$\bot(x)$	
		A	|	$A(x)$	$Human$
		$C \sqcap D$	|	$C(x) \wedge D(x)$	$Human \sqcap Male$
		$C \sqcup D$	|	$C(x) \vee D(x)$	$Nice \sqcup Rich$
		$\neg C$	|	$\neg C(x)$	$\neg Meat$
		$\exists R.C$	|	$\exists y.R(x,y) \wedge C(y)$	$\exists has\_child.Blond$
		$\forall R.C$		$\forall y.R(x,y) \Rightarrow C(y)$	$\forall has\_child.Human$
$C \sqsubseteq D$				$\forall x.C(x) \Rightarrow D(x)$	$Happy\_Father \sqsubseteq Man \sqcap \exists has\_child.Female$
$a{:}C$				$C(a)$	$John{:}Happy\_Father$

DL Semantics

- Semantics is given in terms of an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where
  - $\Delta^{\mathcal{I}}$ is the domain (a non-empty set)
  - $\cdot^{\mathcal{I}}$ is an interpretation function that maps:
    - Concept (class) name A into a subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$
    - Role (property) name R into a subset $R^{\mathcal{I}}$ of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
    - Individual name a into an element of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ s.t. $a^{\mathcal{I}} \neq b^{\mathcal{I}}$ if $a \neq b$ (UNA)
  - Interpretation function $\cdot^{\mathcal{I}}$ is extended to concept expressions:

$$
\begin{array}{rcl}
\top^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} & = & \emptyset \\
(C_1 \sqcap C_2)^{\mathcal{I}} & = & {C_1}^{\mathcal{I}} \cap {C_2}^{\mathcal{I}} \\
(C_1 \sqcup C_2)^{\mathcal{I}} & = & {C_1}^{\mathcal{I}} \cup {C_2}^{\mathcal{I}} \\
(\neg C)^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(\exists R.C)^{\mathcal{I}} & = & \{x \in \Delta^{\mathcal{I}} \mid \exists y.\langle x, y\rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \\
(\forall R.C)^{\mathcal{I}} & = & \{x \in \Delta^{\mathcal{I}} \mid \forall y.\langle x, y\rangle \in R^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}}\}
\end{array}
$$

  - Finally, we say that
    - $\mathcal{I}$ is a model of $C \sqsubseteq D$, written $\mathcal{I} \models C \sqsubseteq D$, iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
    - $\mathcal{I}$ is a model of $a{:}C$, written $\mathcal{I} \models a{:}C$, iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$
    - $\mathcal{I}$ is a model of $(a, b){:}R$, written $\mathcal{I} \models (a, b){:}R$, iff $\langle a^{\mathcal{I}}, b^{\mathcal{I}}\rangle \in R^{\mathcal{I}}$

Note on DL Naming

$\mathcal{AL}$: $C, D \longrightarrow \top \mid \bot \mid A \mid C \sqcap D \mid \neg A \mid \exists R.\top \mid \forall R.C$

$\mathcal{C}$: Concept negation, $\neg C$. Thus, $\mathcal{ALC} = \mathcal{AL} + \mathcal{C}$

$\mathcal{S}$: Used for $\mathcal{ALC}$ with transitive roles $\mathcal{R}_+$

$\mathcal{U}$: Concept disjunction, $C_1 \sqcup C_2$

$\mathcal{E}$: Existential quantification, $\exists R.C$

$\mathcal{H}$: Role inclusion axioms, $R_1 \sqsubseteq R_2$, e.g. $is\_component\_of \sqsubseteq is\_part\_of$

$\mathcal{N}$: Number restrictions, $(\geq n\ R)$ and $(\leq n\ R)$, e.g. $(\geq 3\ has\_Child)$ (has at least 3 children)

$\mathcal{Q}$: Qualified number restrictions, $(\geq n\ R.C)$ and $(\leq n\ R.C)$, e.g. $(\leq 2\ has\_Child.Adult)$ (has at most 2 adult children)

$\mathcal{O}$: Nominals (singleton class), $\{a\}$, e.g. $\exists has\_child.\{mary\}$. **Note**: $a{:}C$ equiv to $\{a\} \sqsubseteq C$ and $(a, b){:}R$ equiv to $\{a\} \sqsubseteq \exists R.\{b\}$

$\mathcal{I}$: Inverse role, R^-, e.g. $isPartOf = hasPart^-$

$\mathcal{F}$: Functional role, f, e.g. $functional(hasAge)$

$\mathcal{R}_+$: transitive role, e.g. $transitive(isPartOf)$

For instance,

$\mathcal{SHIF}$	=	$\mathcal{S} + \mathcal{H} + \mathcal{I} + \mathcal{F} = \mathcal{ALCR_+HIF}$	OWL-Lite
$\mathcal{SHOIN}$	=	$\mathcal{S} + \mathcal{H} + \mathcal{O} + \mathcal{I} + \mathcal{N} = \mathcal{ALCR_+HOIN}$	OWL-DL
$\mathcal{SROIQ}$	=	$\mathcal{S} + \mathcal{R} + \mathcal{O} + \mathcal{I} + \mathcal{Q} = \mathcal{ALCR_+ROIN}$	OWL 2

Semantics of Additional Constructs

- $\mathcal{H}$: Role inclusion axioms, $\mathcal{I} \models R_1 \sqsubseteq R_2$ iff ${R_1}^{\mathcal{I}} \subseteq {R_1}^{\mathcal{I}}$
- $\mathcal{N}$: Number restrictions,
  $(\geq n\ R)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} : |\{y \mid \langle x, y\rangle \in R^{\mathcal{I}}\}| \geq n\}$,
  $(\leq n\ R)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} : |\{y \mid \langle x, y\rangle \in R^{\mathcal{I}}\}| \leq n\}$
- $\mathcal{Q}$: Qualified number restrictions,
  $(\geq n\ R.C)^{\mathcal{I}} = \{x \in |\{y \mid \langle x, y\rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}| \geq n\}$,
  $(\leq n\ R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} : |\{y \mid \langle x, y\rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}| \leq n\}$
- $\mathcal{O}$: Nominals (singleton class), $\{a\}^{\mathcal{I}} = \{a^{\mathcal{I}}\}$
- $\mathcal{I}$: Inverse role, $(R^{-})^{\mathcal{I}} = \{\langle x, y\rangle \mid \langle y, x\rangle \in R^{\mathcal{I}}\}$
- $\mathcal{F}$: Functional role, $I \models fun(f)$ iff $\forall z \forall y \forall z$ if $\langle x, y\rangle \in f^{\mathcal{I}}$ and $\langle x, z\rangle \in f^{\mathcal{I}}$ the $y = z$
- $\mathcal{R}_{+}$: transitive role,
  $(R_{+})^{\mathcal{I}} = \{\langle x, y\rangle \mid \exists z$ such that $\langle x, z\rangle \in R^{\mathcal{I}} \wedge \langle z, y\rangle \in R^{\mathcal{I}}\}$

Basics on Concrete Domains

- Concrete domains: reals, integers, strings, ...

  $(tim, 14)$*:hasAge*
  $(sf, \text{"}SoftComputing\text{"})$*:hasAcronym*
  $(source1, \text{"}ComputerScience\text{"})$*:isAbout*
  $(service2, \text{"}InformationRetrievalTool\text{"})$*:Matches*
  $Minor = Person \sqcap \exists hasAge. \leq_{18}$

- Semantics: a clean separation between "object" classes and concrete domains
  - $D = \langle \Delta_D, \Phi_D \rangle$
  - Δ_D is an interpretation domain
  - Φ_D is the set of concrete domain predicates d with a predefined arity n and fixed interpretation $d^D \subseteq \Delta_D^n$
  - Concrete properties: $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta_D$
- Notation: (D). E.g., $\mathcal{ALC}(D)$ is $\mathcal{ALC}$ + concrete domains

DL Knowledge Base

- A DL Knowledge Base is a pair $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, where
  - $\mathcal{T}$ is a TBox
    - containing general inclusion axioms of the form $C \sqsubseteq D$,
    - concept definitions of the form $A = C$
    - primitive concept definitions of the form $A \sqsubseteq C$
    - role inclusions of the form $R \sqsubseteq P$
    - role equivalence of the form $R = P$
  - $\mathcal{A}$ is a ABox
    - containing assertions of the form $a{:}C$
    - containing assertions of the form $(a, b){:}R$
- An interpretation $\mathcal{I}$ is a model of $\mathcal{K}$, written $\mathcal{I} \models \mathcal{K}$ iff $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$, where
  - $\mathcal{I} \models \mathcal{T}$ ($\mathcal{I}$ is a model of $\mathcal{T}$) iff $\mathcal{I}$ is a model of each element in $\mathcal{T}$
  - $\mathcal{I} \models \mathcal{A}$ ($\mathcal{I}$ is a model of $\mathcal{A}$) iff $\mathcal{I}$ is a model of each element in $\mathcal{A}$

Basic Inference Problems (Formally)

Consistency: Check if knowledge is meaningful

- Is $\mathcal{K}$ satisfiability? $\mapsto$ Is there some model $\mathcal{I}$ of $\mathcal{K}$?
- Is C satisfiability? $\mapsto$ $C^{\mathcal{I}} \neq \emptyset$ for some some model $\mathcal{I}$ of $\mathcal{K}$?

Subsumption: structure knowledge, compute taxonomy

- $\mathcal{K} \models C \sqsubseteq D$? $\mapsto$ Is it true that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for all models $\mathcal{I}$ of $\mathcal{K}$?

Equivalence: check if two classes denote same set of instances

- $\mathcal{K} \models C = D$? $\mapsto$ Is it true that $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all models $\mathcal{I}$ of $\mathcal{K}$?

Instantiation: check if individual a instance of class C

- $\mathcal{K} \models a{:}C$? $\mapsto$ Is it true that $a^{\mathcal{I}} \in C^{\mathcal{I}}$ for all models $\mathcal{I}$ of $\mathcal{K}$?

Retrieval: retrieve set of individuals that instantiate C

- Compute the set $\{a \mid \mathcal{K} \models a{:}C\}$

Reduction to Satisfiability

Problems are all reducible to KB satisfiability

Subsumption: $\mathcal{K} \models C \sqsubseteq D$ iff $\langle \mathcal{T}, \mathcal{A} \cup \{a{:}C \sqcap \neg D\} \rangle$ not satisfiable, where a is a new individual

Equivalence: $\mathcal{K} \models C = D$ iff $\mathcal{K} \models C \sqsubseteq D$ and $\mathcal{K} \models D \sqsubseteq C$

Instantiation: $\mathcal{K} \models a{:}C$ iff $\langle \mathcal{T}, \mathcal{A} \cup \{a{:}\neg C\} \rangle$ not satisfiable

Retrieval: The computation of the set $\{a \mid \mathcal{K} \models a{:}C\}$ is reducible to the instance checking problem

Reasoning in DLs: Basics

- OWL 2: tableaux based algorithms
- OWL 2 EL: structural based algorithms
- OWL 2 QL: query rewriting based algorithms
- OWL 2 RL: logic programming based algorithms

Tableaux: Basics

- Tableaux algorithm deciding satisfiability
- Try to build a tree-like model $\mathcal{I}$ of the KB
- Decompose concepts C syntactically
  - Apply tableau expansion rules
  - Infer constraints on elements of model
- Tableau rules correspond to constructors in logic ($\sqcap, \sqcup, \ldots$)
  - Some rules are nondeterministic (e.g., $\sqcup, \leq$)
  - In practice, this means search
- Stop when no more rules applicable or clash occurs
  - Clash is an obvious contradiction, e.g., $A(x), \neg A(x)$
- Cycle check (blocking) may be needed for termination

Negation Normal Form (NNF)

- We have to transform concepts into Negation Normal Form: push negation inside using de Morgan' laws

$$
\begin{array}{rcl}
\neg\top & \mapsto & \bot \\
\neg\bot & \mapsto & \top \\
\neg\neg C & \mapsto & C \\
\neg(C_1 \sqcap C_2) & \mapsto & \neg C_1 \sqcup \neg C_2 \\
\neg(C_1 \sqcup C_2) & \mapsto & \neg C_1 \sqcap \neg C_2
\end{array}
$$

and

$$
\begin{array}{rcl}
\neg(\exists R.C) & \mapsto & \forall R.\neg C \\
\neg(\forall R.C) & \mapsto & \exists R.\neg C
\end{array}
$$

Completion-Forest

- This is a forest of trees, where
  - each node x is labelled with a set $\mathcal{L}(x)$ of concepts
  - each edge $\langle x, y \rangle$ is labelled with $\mathcal{L}(\langle x, y \rangle) = \{R\}$ for some role R (edges correspond to relationships between pairs of individuals)
- The forest is initialized with
  - a root node a, labelled $\mathcal{L}(x) = \emptyset$ for each individual a occurring in the KB
  - an edge $\langle a, b \rangle$ labelled $\mathcal{L}(\langle a, b \rangle) = \{R\}$ for each (a, b):R occurring in the KB
- Then, for each a:C occurring in the KB, set $\mathcal{L}(a) \to \mathcal{L}(a) \cup \{C\}$
- The algorithm expands the tree either by extending $\mathcal{L}(x)$ for some node x or by adding new leaf nodes.
- Edges are added when expanding $\exists R.C$
- A completion-forest contains a clash if, for a node x, $\{C, \neg C\} \subseteq \mathcal{L}(x)$
- If nodes x and y are connected by an edge$\langle x, y \rangle$, then y is called a successor of x and x is called a predecessor of y. Ancestor is the transitive closure of predecessor.
- A node y is called an R-successor of a node x if y is a successor of x and $\mathcal{L}(\langle x, y \rangle) = \{R\}$.
- The algorithm returns “satisfiable" if rules can be applied s.t. they yield a clash-free, complete (no more rules can be applied) completion forest

$\mathcal{ALC}$ Tableau rules without GCI's

Rule		**Description**
$(\sqcap)$	if 1.	$C_1 \sqcap C_2 \in \mathcal{L}(x)$ and
	2.	$\{C_1, C_2\} \not\subseteq \mathcal{L}(x)$
	then	$\mathcal{L}(x) \to \mathcal{L}(x) \cup \{C_1, C_2\}$
$(\sqcup)$	if 1.	$C_1 \sqcup C_2 \in \mathcal{L}(x)$ and
	2.	$\{C_1, C_2\} \cap \mathcal{L}(x) = \emptyset$
	then	$\mathcal{L}(x) \to \mathcal{L}(x) \cup \{C\}$ for some $C \in \{C_1, C_2\}$
$(\exists)$	if 1.	$\exists R.C \in \mathcal{L}(x)$ and
	2.	x has no R-successor y with $C \in \mathcal{L}(y)$
	then	create a new node y with $\mathcal{L}(\langle x, y\rangle) = \{R\}$ and $\mathcal{L}(y) = \{C\}$
$(\forall)$	if 1.	$\forall R.C \in \mathcal{L}(x)$ and
	2.	x has an R-successor y with $C \notin \mathcal{L}(y)$
	then	$\mathcal{L}(y) \to \mathcal{L}(y) \cup \{C\}$

Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D\}$$

x

Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D\}$$

x

Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D\}$$

$$x$$

Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

x

Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

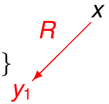
x

Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

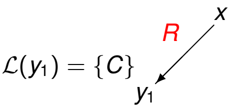
$$\mathcal{L}(y_1) = \{C\}$$



Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

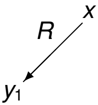


Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

$$\mathcal{L}(y_1) = \{C, \neg C \sqcup \neg D\}$$

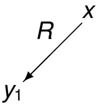


Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

$$\mathcal{L}(y_1) = \{C, \neg C \sqcup \neg D\}$$

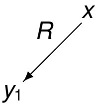


Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

$$\mathcal{L}(y_1) = \{C, \neg C \sqcup \neg D, \neg C\}$$

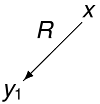


Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

$$\mathcal{L}(y_1) = \{C, \neg C \sqcup \neg D, \neg C\}$$



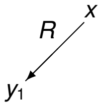
Clash

Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

$$\mathcal{L}(y_1) = \{C, \neg C \sqcup \neg D\}$$

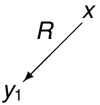


Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

$$\mathcal{L}(y_1) = \{C, \neg C \sqcup \neg D, \neg D\}$$

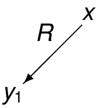


Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

$$\mathcal{L}(y_1) = \{C, \neg C \sqcup \neg D, \neg D\}$$

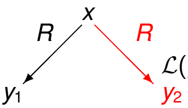


Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

$$\mathcal{L}(y_1) = \{C, \neg C \sqcup \neg D, \neg D\}$$



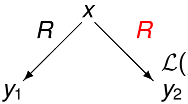
$$\mathcal{L}(y_2) = \{D\}$$

Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

$$\mathcal{L}(y_1) = \{C, \neg C \sqcup \neg D, \neg D\}$$



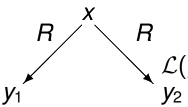
$$\mathcal{L}(y_2) = \{D\}$$

Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

$\mathcal{L}(y_1) = \{C, \neg C \sqcup \neg D, \neg D\}$



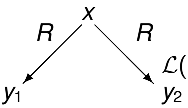
$\mathcal{L}(y_2) = \{D, \neg C \sqcup \neg D\}$

Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes. w

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

$\mathcal{L}(y_1) = \{C, \neg C \sqcup \neg D, \neg D\}$



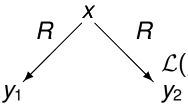
$\mathcal{L}(y_2) = \{D, \neg C \sqcup \neg D\}$

Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

$\mathcal{L}(y_1) = \{C, \neg C \sqcup \neg D, \neg D\}$



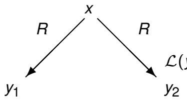
$\mathcal{L}(y_2) = \{D, \neg C \sqcup \neg D, \neg C\}$

Example

Is $\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D$ satisfiable? Yes.

$$\mathcal{L}(x) = \{\exists R.C, \forall R.(\neg C \sqcup \neg D), \exists R.D\}$$

$\mathcal{L}(y_1) = \{C, \neg C \sqcup \neg D, \neg D\}$



$\mathcal{L}(y_2) = \{D, \neg C \sqcup \neg D, \neg C\}$

- Finished. No more rules applicable and the tableau is complete and clash-free
- Hence, the concept is satisfiable
- The tree corresponds to a model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$
  - The nodes are the elements of the domain: $\Delta^{\mathcal{I}} = \{x, y_1, y_2\}$
  - For each atomic concept A, set $A^{\mathcal{I}} = \{z \mid A \in \mathcal{L}(z)\}$
    - $C^{\mathcal{I}} = \{y_1\}, D^{\mathcal{I}} = \{y_2\}$
  - For each role R, set $R^{\mathcal{I}} = \{\langle x, y \rangle \mid$ there is an edge labeled R from x to $y\}$
    - $R^{\mathcal{I}} = \{\langle x, y_1 \rangle, \langle x, y_2 \rangle\}$
  - It can be shown that $x \in (\exists R.C \sqcap \forall R.(\neg C \sqcup \neg D) \sqcap \exists R.D)^{\mathcal{I}} \neq \emptyset$

Soundness and Completeness

Theorem

Let $\mathcal{A}$ be an $\mathcal{ALC}$ ABox and F a completion-forest obtained by applying the tableau rules to $\mathcal{A}$. Then

1. *The rule application terminates;*
2. *If F is clash-free and complete, then F defines a (canonical) (tree) model for $\mathcal{A}$; and*
3. *If $\mathcal{A}$ has a model $\mathcal{I}$, then the rules can be applied such that they yield a clash-free and complete completion-forest.*

KBs with GCIs

- We have seen how to test the satisfiability of an ABox $\mathcal{A}$
- But, how can we check if a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ is satisfiable with $\mathcal{T} \neq \emptyset$?
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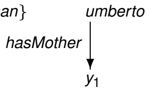

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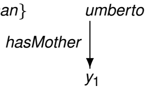

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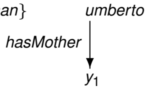

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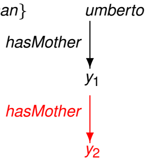

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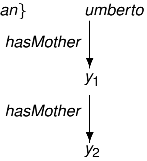

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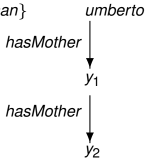

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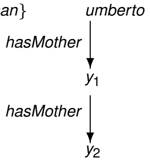

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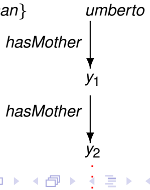

Node Blocking in $\mathcal{ALC}$

- When creating new node, check ancestors for equal label set
- If such a node is found, new node is blocked
- No rule is applied to blocked nodes

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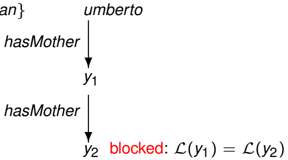

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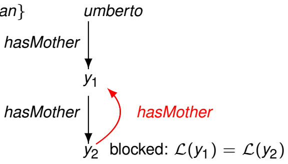


- Block represents cyclical model
  - $\Delta^{\mathcal{I}} = \{umberto, y_1, y_2\}$
  - $Human^{\mathcal{I}} = \{umberto, y_1, y_2\}$
  - $hasMother^{\mathcal{I}} = \{\langle umberto, y_1\rangle, \langle y_1, y_2\rangle, \langle y_2, y_1\rangle\}$

Blocking in $\mathcal{ALC}$

- A non-root node x is blocked if for some ancestor y, y is blocked or $\mathcal{L}(x) = \mathcal{L}(y)$, where y is not a root node
- A blocked node x is indirectly blocked if its predecessor is blocked, otherwise it is directly blocked
- If x is directly blocked, it has a unique ancestor y such that $\mathcal{L}(x) = \mathcal{L}(y)$
- if there existed another ancestor z such that $\mathcal{L}(x) = \mathcal{L}(z)$ then either y or z must be blocked
- If x is directly blocked and y is the unique ancestor such that $\mathcal{L}(x) = \mathcal{L}(y)$, we will say that y blocks x

$\mathcal{ALC}$ Tableau rules with GCI's

Rule		**Description**
$(\sqcap)$	if 1.	$C_1 \sqcap C_2 \in \mathcal{L}(x)$, x is not indirectly blocked and
	2.	$\{C_1, C_2\} \not\subseteq \mathcal{L}(x)$
	then	$\mathcal{L}(x) \to \mathcal{L}(x) \cup \{C_1, C_2\}$
$(\sqcup)$	if 1.	$C_1 \sqcup C_2 \in \mathcal{L}(x)$, x is not indirectly blocked and
	2.	$\{C_1, C_2\} \cap \mathcal{L}(x) = \emptyset$
	then	$\mathcal{L}(x) \to \mathcal{L}(x) \cup \{C\}$ for some $C \in \{C_1, C_2\}$
$(\exists)$	if 1.	$\exists R.C \in \mathcal{L}(x)$, x is not blocked and
	2.	x has no R-successor y with $C \in \mathcal{L}(y)$
	then	create a new node y with $\mathcal{L}(\langle x, y\rangle) = \{R\}$ and $\mathcal{L}(y) = \{C\}$
$(\forall)$	if 1.	$\forall R.C \in \mathcal{L}(x)$, x is not indirectly blocked and
	2.	x has an R-successor y with $C \notin \mathcal{L}(y)$
	then	$\mathcal{L}(y) \to \mathcal{L}(y) \cup \{C\}$
$(\sqsubseteq)$	if 1.	$C \sqsubseteq D \in \mathcal{T}$, x is not indirectly blocked and
	2.	$\{nnf(\neg C), D\} \cap \mathcal{L}(x) = \emptyset$
	then	$\mathcal{L}(x) \to \mathcal{L}(x) \cup \{E\}$ for some $E \in \{nnf(\neg C), D\}$ ($nnf(\neg C)$ is NNF of $\neg C$)

Soundness and Completeness

Theorem

Let $\mathcal{K}$ be an $\mathcal{ALC}$ KB and F a completion-forest obtained by applying the tableau rules to $\mathcal{K}$. Then

1. *The rule application terminates;*
2. *If F is clash-free and complete, then F defines a (canonical) (tree) model for $\mathcal{K}$; and*
3. *If $\mathcal{K}$ has a model $\mathcal{I}$, then the rules can be applied such that they yield a clash-free and complete completion-forest.*

Fuzzy DLs Basics

- We have seen how to "fuzzify" classical sets and FOL
  - Fuzzy statements are of the form $\langle \phi, n \rangle$, where ϕ is a statement and $n \in [0, 1]$
- The natural extension to fuzzy DLs consists then in replacing ϕ with a DL expression
- Several fuzzy variants of DLs have been proposed: they can be classified according to
  - The DL resp. ontology language that they generalize
  - The allowed fuzzy constructs
  - The underlying fuzzy logic
  - Their reasoning algorithms and computational complexity results

- In classical DLs, a concept C is interpreted by an interpretation $\mathcal{I}$ as a set of individuals
- In fuzzy DLs, a concept C is interpreted by $\mathcal{I}$ as a fuzzy set of individuals
- Each individual is instance of a concept to a degree in $[0, 1]$
- Each pair of individuals is instance of a role to a degree in $[0, 1]$

- $\langle a{:}C, n\rangle$ states that a is an instance of concept/class C with degree at least n
- $\langle (a, b){:}R, n\rangle$ states that $\langle a, b\rangle$ is an instance of relation R with degree at least n
- $\langle C_1 \sqsubseteq C_2, n\rangle$ states a vague subsumption relationship
  - The FOL statement $\forall x.C_1(x) \rightarrow C_2(x)$ is true to degree at least n
- Note: one may find also fuzzy DL expressions $\langle \alpha \geq n\rangle$, $\langle \alpha \leq n\rangle$, $\langle \alpha > n\rangle$, $\langle \alpha < n\rangle$, and $\langle \alpha = n\rangle$
- We use the form $\langle \alpha, n\rangle$, i.e. $\langle \alpha \geq n\rangle$ only
  - Remind that graded axioms are intended to be produced semi- or automatically
  - Hardly they may have the form $\langle \alpha \leq n\rangle$, $\langle \alpha > n\rangle$ or $\langle \alpha < n\rangle$

The semantics is an immediate consequence of the First-Order-Logic translation of DLs expressions

Interpretation:

$$\mathcal{I} = \Delta^{\mathcal{I}}$$
$$C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$$
$$R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$$

$\otimes$ = t-norm
$\oplus$ = s-norm
$\neg$ = negation
$\Rightarrow$ = implication

Concepts:

Syntax	Semantics		
$C, D \longrightarrow \top \mid$	$\top^{\mathcal{I}}(x)$	$=$	1
$\bot \mid$	$\bot^{\mathcal{I}}(x)$	$=$	0
$A \mid$	$A^{\mathcal{I}}(x)$	$\in$	$[0,1]$
$C \sqcap D \mid$	$(C_1 \sqcap C_2)^{\mathcal{I}}(x)$	$=$	$C_1{}^{\mathcal{I}}(x) \otimes C_2{}^{\mathcal{I}}(x)$
$C \sqcup D \mid$	$(C_1 \sqcup C_2)^{\mathcal{I}}(x)$	$=$	$C_1{}^{\mathcal{I}}(x) \oplus C_2{}^{\mathcal{I}}(x)$
$C \to D \mid$	$(C \to D)^{\mathcal{I}}(x)$	$=$	$C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$
$\neg C \mid$	$(\neg C)^{\mathcal{I}}(x)$	$=$	$\neg C^{\mathcal{I}}(x)$
$\exists R.C \mid$	$(\exists R.C)^{\mathcal{I}}(x)$	$=$	$\sup_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)$
$\forall R.C$	$(\forall R.C)^{\mathcal{I}}(x)$	$=$	$\inf_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)\}$
$\{a\}$	$\{a\}^{\mathcal{I}}(x)$	$=$	1 if $a^{\mathcal{I}} = x$, else 0

Assertions: $\langle a{:}C, r\rangle, \mathcal{I} \models \langle a{:}C, r\rangle$ iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq r$ (similarly for roles)

General Inclusion Axioms: $\langle C \sqsubseteq D, r\rangle$,

- $\mathcal{I} \models \langle C \sqsubseteq D, r\rangle$ iff $\inf_{x \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq r$

Some Remarks

- Like for fuzzy FOL, $\forall$ and $\exists$ are not complementary in general: i.e. $\forall R.C \not\equiv \neg\exists R.\neg C$
- $\forall R.C \equiv \neg\exists R.\neg C$ under Łukasiewicz logic and SFL
- $\langle C \sqsubseteq D, n\rangle$ may be rewritten as $\langle \top \sqsubseteq C \rightarrow D, n\rangle$
- In early works, a fuzzy GCI is of the form $C \sqsubseteq D$ with semantics:
  - $\mathcal{I}$ is a model of $C \sqsubseteq D$ iff for every $x \in \Delta^{\mathcal{I}}$ we have that $C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x)$
  - This is the same of fuzzy axiom $\langle \top \sqsubseteq C \rightarrow_x D, 1\rangle$, where $\rightarrow_x$ is an r-implication
- Disjointness: use $\langle C \sqcap D \sqsubseteq \bot, 1\rangle$ rather than $\langle C \sqsubseteq \neg D, 1\rangle$
  - they are not the same, e.g. $\langle A \sqsubseteq \neg A, 1\rangle$ says that $A^{\mathcal{I}}(x) \leq 0.5$, for all $\mathcal{I}$ and for all $x \in \Delta^{\mathcal{I}}$

Witnessed Interpretation

- Witnessed Interpretation:
  - Infima and suprema are attained at some point

$$
\begin{array}{rcl}
(\exists R.C)^{\mathcal{I}}(x) & = & R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y) \text{ for some } y \in \Delta^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}}(x) & = & R^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y) \text{ for some } y \in \Delta^{\mathcal{I}} \\
(C \sqsubseteq D)^{\mathcal{I}} & = & C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \text{ for some } x \in \Delta^{\mathcal{I}}
\end{array}
$$

- It is customary to stick to witnessed interpretations only

- Fuzzy knowledge base: $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$
  - $\mathcal{T}$ is a fuzzy TBox, that is a finite set of fuzzy GCI
  - $\mathcal{A}$ is a *fuzzy ABox*, that is a finite set of fuzzy assertions
- Acyclic fuzzy ontologies: TBox with axioms of the form

$$\begin{array}{lll} A & \sqsubseteq_n & C \text{ (primitive GCI)} \\ A & \tilde{\sqsubseteq} & C \text{ (primitive GCI)} \\ A & \tilde{=} & C \text{ (definitional GCI)} \end{array}$$

  - A concept name
  - $A \sqsubseteq_n C$ shorthand for $\langle \top \sqsubseteq A \rightarrow C, n \rangle$
  - No nominal $\{a\}$ occurs in the TBox

- We say that
  - concept name A directly uses concept name B w.r.t. $\mathcal{T}$, denoted $A \rightarrow_{\mathcal{T}} B$, if A is the head of some axiom $\tau \in \mathcal{T}$ such that B occurs in the body of τ
  - concept name A uses concept name B w.r.t. $\mathcal{T}$, denoted $A \rightsquigarrow_{\mathcal{T}} B$, if there exist concept names $A_1, \ldots, A_n$, such that $A_1 = A$, $A_n = B$ and, for every $1 \leq i < n$, it holds that $A_i \rightarrow_{\mathcal{T}} A_{i+1}$
- TBox $\mathcal{T}$ is cyclic (acyclic) if there is (no) A such that $A \rightsquigarrow_{\mathcal{T}} A$
- TBox $\mathcal{T}$ is unfoldable if
  - $\mathcal{T}$ is acyclic
  - If $A \tilde{=} C \in \mathcal{T}$ then A does not occur in the head of any other axiom

- $\mathcal{I}$ satisfies (is a model of) $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ iff it satisfies each element in $\mathcal{A}$ and $\mathcal{T}$
- A fuzzy KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ entails an axiom E, denoted $\mathcal{K} \models E$, iff every model of $\mathcal{K}$ satisfies E
- We say that two concepts C and D are equivalent, denoted $C \equiv_{\mathcal{K}} D$ iff in every model $\mathcal{I}$ of $\mathcal{K}$ and for all $x \in \Delta^{\mathcal{I}}$, $C^{\mathcal{I}}(x) = D^{\mathcal{I}}(x)$
- Best entailment degree: for assertion of GCI ϕ

$$bed(\mathcal{K}, \phi) = \sup \{r \mid \mathcal{K} \models \langle \phi, r \rangle\}$$

- Best satisfiability degree: for concept C

$$bsd(\mathcal{K}, C) = \sup_{\mathcal{I} \models \mathcal{K}} \sup_{x \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \; .$$

Some Salient Fuzzy Concept Equivalences

Property	Łukasiewicz	Gödel	Product	SFL
$C \sqcap \neg C \equiv \bot$	•	•	•	
$C \sqcup \neg C \equiv \top$	•			
$C \sqcap C \equiv C$		•		•
$C \sqcup C \equiv C$		•		•
$\neg\neg C \equiv C$	•			•
$C \to D \equiv \neg C \sqcup D$	•			•
$C \to D \equiv \neg D \to \neg C$	•			•
$\neg(C \to D) \equiv C \sqcap \neg D$	•			•
$\neg(C \sqcap D) \equiv \neg C \sqcup \neg D$	•	•	•	•
$\neg(C \sqcup D) \equiv \neg C \sqcap \neg D$	•	•	•	•
$C \sqcap (D \sqcup E) \equiv (C \sqcap D) \sqcup (C \sqcap E)$		•		•
$C \sqcup (D \sqcap E) \equiv (C \sqcup D) \sqcap (C \sqcup E)$		•		•
$\exists R.C \equiv \neg\forall R.\neg C$	•			•

Towards Fuzzy OWL 2 and its Profiles

- Recall that OWL 2 relates to $\mathcal{SROIQ}(D)$
- We need to extend the semantics to fuzzy $\mathcal{SROIQ}(D)$
- Additionally, we add
  - modifiers (e.g., *very*)
  - concrete fuzzy concepts (e.g., *Young*)
  - both additions have explicit membership functions
  - other extensions:
    - aggregation functions: weighted sum, OWA, fuzzy integrals
    - fuzzy rough sets, fuzzy spatial, fuzzy numbers

Number Restrictions, Inverse, Transitive roles, ...

- The semantics of the concept $(\geq n\ R.C)$ is: $\wedge$ interpreted as Gödel t-norm

$$\exists y_1, \ldots, y_n. \bigwedge_{i=1}^{n} R(x, y_i) \wedge C(y_i) \wedge \bigwedge_{1 \leq i < j \leq n} y_i \neq y_j$$

- The semantics of the concept $(\leq n\ R.C)$ is: $\wedge$ interpreted as Gödel t-norm

$$(\leq n\ R)^{\mathcal{I}}(x) = \forall y_1, \ldots, y_{n+1}. \bigwedge_{i=1}^{n+1} (R(x, y_i) \wedge C(y_i)) \Rightarrow \bigvee_{1 \leq i < j \leq n+1} y_i = y_j$$

- Note: $(\geq 1\ R) \equiv \exists R.\top$
- For transitive roles R we impose: for all $x, y \in \Delta^{\mathcal{I}}$

$$R^{\mathcal{I}}(x, y) \geq \sup_{z \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x, z) \otimes R^{\mathcal{I}}(z, y)$$

- For inverse roles we have for all $x, y \in \Delta^{\mathcal{I}}$

$$R^{\mathcal{I}}(x, y) = R^{\mathcal{I}}(y, x)$$

- The semantics of fucntional roles $fun(R)$ is

$$\forall x \forall y \forall z.\ R(x, y) \wedge R(x, z) \Rightarrow y = z$$

- Similar for other $\mathcal{SROIQ}$ constructs

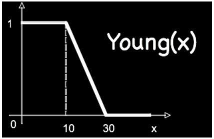
Fuzzy Concrete Domains

- E.g., *Small*, *Young*, *High*, *etc*. with explicit membership function
- Use the idea of concrete domains:
  - $D = \langle \Delta_D, \Phi_D \rangle$
  - Δ_D is an interpretation domain
  - Φ_D is the set of concrete unary fuzzy domain predicates d and fixed interpretation $d^D \colon \Delta_D \to [0, 1]$
- Specifically,

$$\begin{array}{lcl} \mathbf{d} & \to & ls(a,b) \mid rs(a,b) \mid tri(a,b,c) \mid trz(a,b,c,d) \\ & & \mid \ \geq_v \ \mid \ \leq_v \ \mid \ =_v \end{array}$$

$$C, D \ \to \ \forall T.\mathbf{d} \mid \exists T.\mathbf{d}$$

- Representation of Young Person:



$$
\begin{aligned}
Minor &= Person \sqcap \exists hasAge. \leq_{18} \\
YoungPerson &= Person \sqcap \exists hasAge.ls(10, 30)
\end{aligned}
$$

- Representation of Heavy Rain:

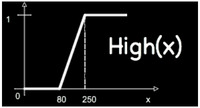
$$HeavyRain = Rain \sqcap \exists hasPrecipitationRate.rs(5, 7.5)$$

Modifiers

- *Very*, *moreOrLess*, *slightly*, etc.
- Fuzzy modifier m with function $f_m \colon [0,1] \to [0,1]$

$$C \quad \to \quad m(C) \mid \forall T.m(\mathbf{d}) \mid \exists T.m(\mathbf{d})$$

  where m is a linear modifier
- Representation of Sport Car



$$SportsCar \quad = \quad Car \sqcap \exists speed.very(rs(80,250))$$

- Representation of Very Heavy Rain

$$VeryHeavyRain = Rain \sqcap \exists hasPrecipitationRate.very(rs(5,7.5)) \, .$$

Aggregation Operators

- Aggregation operators: aggregate concepts, using functions such as the mean, median, weighted sum operators
- Given an n-ary aggregation operator $@ : [0,1]^n \to [0,1]$
  - We fuzzy concepts by allowing to apply @ to n concepts $C_1, \ldots, C_n$, i.e.

$$C \quad \to \quad @(C_1, \ldots, C_n)$$

- Semantics:

$$@(C_1, \ldots, C_n)^{\mathcal{I}}(x) \quad = \quad @(C_1^{\mathcal{I}}(x), \ldots, C_n^{\mathcal{I}}(x)) \ .$$

- Allows to express the concept

$$\mathit{GoodHotel} = 0.3 \cdot \mathit{ExpensiveHotel} + 0.7 \cdot \mathit{LuxuriousHotel}$$

  - The membership function of good hotels is the weighted sum of being an expensive and luxurious hotel

Some Applications

- Information retrieval
- Recommendation systems
- Image interpretation
- Ambient intelligence
- Ontology merging
- Matchmaking
- decision making
- Summarization
- Robotics perception
- Software design
- Machine learning

Example (Graded Entailment)

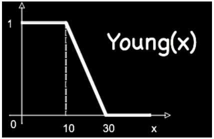
		
audi_tt	*mg*	*ferrari_enzo*

Car	*speed*
audi_tt	243
mg	≤ 170
ferrari_enzo	≥ 350

$$\mathit{SportsCar} \quad = \quad \mathit{Car} \sqcap \exists \mathit{hasSpeed}.\mathit{very}(\mathit{High})$$

$$\begin{array}{lll} \mathcal{K} & \models & \langle \mathit{ferrari\_enzo}{:}\mathit{SportsCar}, 1\rangle \\ \mathcal{K} & \models & \langle \mathit{audi\_tt}{:}\mathit{SportsCar}, 0.92\rangle \\ \mathcal{K} & \models & \langle \mathit{mg}{:}\neg \mathit{SportsCar}, 0.72\rangle \end{array}$$

Example (Graded Subsumption)

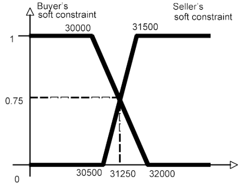


$$
\begin{array}{rcl}
\mathit{Minor} & = & \mathit{Person} \sqcap \exists \mathit{hasAge}. \leq_{18} \\
\mathit{YoungPerson} & = & \mathit{Person} \sqcap \exists \mathit{hasAge}. \mathit{Young} \\
 & & \mathit{fun}(\mathit{hasAge})
\end{array}
$$

$$\mathcal{K} \models \langle \mathit{Minor} \sqsubseteq \mathit{YoungPerson}, 0.6 \rangle$$

Note: without an explicit membership function of *Young*, this inference cannot be drawn

Example (Simplified Matchmaking)



- A car seller sells an Audi TT for 31500 €, as from the catalog price.
- A buyer is looking for a sports-car, but wants to to pay not more than around 30000 €
- Classical sets: the problem relies on the crisp conditions on price
- More fine grained approach: to consider prices as fuzzy sets (as usual in negotiation)
  - Seller may consider optimal to sell above 31500 €, but can go down to 30500 €
  - The buyer prefers to spend less than 30000 €, but can go up to 32000 €

    $$AudiTT = SportsCar \sqcap \exists hasPrice.rs(30500, 31500)$$
    $$Query = SportsCar \sqcap \exists hasPrice.ls(30000, 32000)$$
  - Highest degree to which the concept $C = AudiTT \sqcap Query$ is satisfiable is 0.75 (the degree to which the Audi TT and the query matches is 0.75)
  - The car may be sold at 31250 €

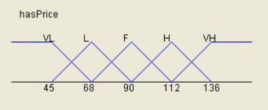
Example: Learning fuzzy GCIs from data

- Learning of fuzzy GCIs from crisp data
- Use Case: What are Good hotels, using TripAdvisor data?
  - Given
    - OWL 2 Ontology about meaningful city entities and their descriptions
    - TripAdvisor data about hotels and user judgments
  - We may learn that in e.g., Pisa, Italy

$$\langle \exists hasAmenity.Babysitting \sqcap \exists hasPrice.fair \sqsubseteq Good\_Hotel, 0.282 \rangle$$

"A hotel having babysitting as amenity and a fair price is a good hotel (to degree 0.282)"

  - Real valued price attribute *hasPrice* has been automatically fuzzyfied



Example: Multi-Criteria Decision Making

- We have to select among two sites, A_1 , A_2
- There are two criteria (C_1 -Transportation Issues, and C_2 -Public Nuisance) for judgement
- There are two experts (E_1 , E_2) that make judgments
- The decision matrix of the experts is shown below:

E_1		Criteria	
		0.48	0.52
Alter.		C_1	C_2
x_1	A_1	$tri(0.6, 0.7, 0.8)$	$tri(0.9, 0.95, 1.0)$
x_2	A_2	$tri(0.6, 0.7, 0.8)$	$tri(0.4, 0.5, 0.6)$

E_2		Criteria	
		0.52	0.48
Alter.		C_1	C_2
x_1	A_1	$tri(0.55, 0.6, 0.7)$	$tri(0.4, 0.45, 0.5)$
x_2	A_2	$tri(0.35, 0.4, 0.45)$	$tri(0.5, 0.55, 0.6)$

- For each expert $k = 1, 2$, for each alternative $i = 1, 2$ and for each criteria $j = 1, 2$, we define the concept

$$P^k_{ij} = \exists \text{hasScore}.a^k_{ij}$$

- Now, for each expert k and alternative i, we define the weighted concept

$$A^k_i = w^k_1 \cdot P^k_{i1} + w^k_2 \cdot P^k_{i2}$$

- Finally, we combine the two experts outcome, by defining the weighted concept

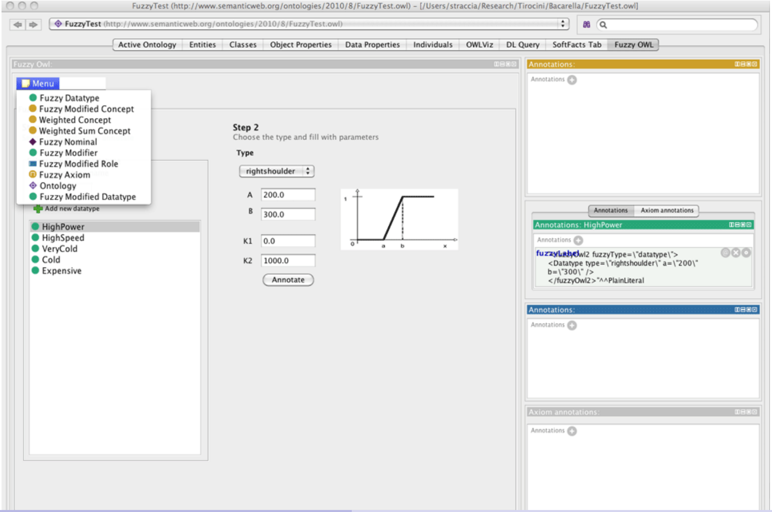
$$A_i = 0.5 \cdot A^1_i + 0.5 \cdot A^2_i$$

- It can be verified that $rv(\mathcal{K}, A_1) = bsd(\mathcal{K}, A_1) = 0.26$ and $rv(\mathcal{K}, A_2) = bsd(\mathcal{K}, A_2) = 0.37$

Representing Fuzzy OWL Ontologies in OWL

- OWL 2 is W3C standard, with classical logic semantics
  - Hence, cannot support natively Fuzzy Logic
- However, Fuzzy OWL 2, has been defined using OWL 2
  - Uses the axiom annotation feature of OWL 2
- Any Fuzzy OWL 2 ontology is a legal OWL 2 ontology

- A java parser for Fuzzy OWL 2 exists
- Protégé plug-in exists to encode Fuzzy OWL ontologies



Reasoning Problems and Algorithms

Consistency problem:

- Is $\mathcal{K}$ satisfiable?
- Is C coherent, i.e. is $C^{\mathcal{I}}(x) > 0$ for some $\mathcal{I} \models \mathcal{K}$ and $x \in \Delta^{\mathcal{I}}$?

Instance checking problem:

- Does $\mathcal{K} \models \langle a{:}C, n \rangle$ hold?

Subsumption problem:

- Does $\mathcal{K} \models \langle C \sqsubseteq D, n \rangle$ hold?

Best entailment degree problem:

- What is $bed(\mathcal{K}, \phi)$?

Best satisfiability degree problem:

- What is $bsd(\mathcal{K}, \phi)$?

Instance retrieval problem:

- Compute the set $\{\langle a, n \rangle \mid n = bed(\mathcal{K}, a{:}C)\}$

Top-k retrieval problem:

- Compute the top-k ranked elements of $\{\langle a, n \rangle \mid n = bed(\mathcal{K}, a{:}C)\}$

Some Reductions

- $\mathcal{K}$ is satisfiable iff $bsd(\mathcal{K}, a{:}\bot) > 0$, where a is a new individual.
- C is coherent w.r.t. $\mathcal{K}$ if one of the following holds:
  - $\mathcal{K} \cup \{\langle a{:}C > 0\rangle\}$ is satisfiable, where a is a new individual
  - $\mathcal{K} \not\models \langle C \sqsubseteq \bot, 1\rangle$
  - $bsd(\mathcal{K}, C) > 0$
- $\mathcal{K} \models \langle a{:}C, n\rangle$ if one of the following holds:
  - $\mathcal{K} \cup \{\langle a{:}C < n\rangle\}$ is not satisfiable
  - $bed(\mathcal{K}, a{:}C) \geq n$
- $\mathcal{K} \models \langle C \sqsubseteq D, n\rangle$ if one of the following holds:
  - $\mathcal{K} \cup \{\langle a{:}C \rightarrow D < n\rangle\}$ is not satisfiable, where a is a new individual
  - $bed(\mathcal{K}, C \sqsubseteq D) \geq n$
- We have that

$$
\begin{aligned}
bed(\mathcal{K}, \phi) &= \min x.\ \text{such that } \mathcal{K} \cup \{\langle \phi \leq x\rangle\} \text{ satisfiable} \\
bsd(\mathcal{K}, \phi) &= \max x.\ \text{such that } \mathcal{K} \cup \{\langle \phi \geq x\rangle\} \text{ satisfiable}
\end{aligned}
$$

Reasoning in Fuzzy DLs: Basics

- Algorithms for fuzzy DLs: are a mixture of classical DLs reasoning algorithms and algorithms for Mathematical Fuzzy Logic
- Fuzzy OWL 2:
  - Fuzzy tableaux based algorithms
    - Tableaux and non deterministic tableaux
    - Operational Research
  - Reduction into classical DLs
- Fuzzy OWL 2 EL: fuzzy structural based algorithms
- Fuzzy OWL 2 QL: fuzzy query rewriting based algorithms
- fuzzy OWL 2 RL: fuzzy logic programming based algorithms

OR Fuzzy Tableaux: $\mathcal{ALC}$ under SFL over $[0, 1]$

- Works as for classical $\mathcal{ALC}$ on completion forests
  - Blocking is as for classical $\mathcal{ALC}$
  - The completion forest is expanded by repeatedly applying inference rules
  - The completion-forest is complete when none of the rules are applicable
- Additionally, at each inference step we add equational constraints that have to hold
- Eventually, the initial KB is satisfiable if the final set of equational constraints has a solution
  - For the latter case, we may use a MILP solver

Rule		Description
(var)		For variable $x_{v:C}$ add $x_{v:C} \in [0,1]$ to $\mathcal{C}_{\mathcal{F}}$. For variable $x_{(v,w):R}$, add $x_{(v,w):R} \in [0,1]$ to $\mathcal{C}_{\mathcal{F}}$
$(\bar{A})$	if	$\neg A \in \mathcal{L}(v)$ then add $x_{v:A} = 1 - x_{v:\neg A}$ to $\mathcal{C}_{\mathcal{F}}$
$(\bot)$	If	$\bot \in \mathcal{L}(v)$ then add $x_{v:\bot} = 0$ to $\mathcal{C}_{\mathcal{F}}$
$(\top)$	If	$\top \in \mathcal{L}(v)$ then add $x_{v:\top} = 1$ to $\mathcal{C}_{\mathcal{F}}$
$(\sqcap)$	if	$C_1 \sqcap C_2 \in \mathcal{L}(v)$, v is not indirectly blocked
	then	$\mathcal{L}(v) \to \mathcal{L}(v) \cup \{C_1, C_2\}$, and add $x_{v:C_1} \otimes x_{v:C_2} \geq x_{v:C_1 \sqcap C_2}$ to $\mathcal{C}_{\mathcal{F}}$
$(\sqcup)$	if	$C_1 \sqcup C_2 \in \mathcal{L}(v)$, v is not indirectly blocked
	then	$\mathcal{L}(v) \to \mathcal{L}(v) \cup \{C_1, C_2\}$, and add $x_{v:C_1} \oplus x_{v:C_2} \geq x_{v:C_1 \sqcup C_2}$ to $\mathcal{C}_{\mathcal{F}}$
$(\forall)$	if	$\forall R.C \in \mathcal{L}(v)$, v is not indirectly blocked
	then	$\mathcal{L}(w) \to \mathcal{L}(w) \cup \{C\}$, and add $x_{w:C} \geq x_{v:\forall R.C} \otimes x_{(v,w):R}$ to $\mathcal{C}_{\mathcal{F}}$
$(\exists)$	if	$\exists R.C \in \mathcal{L}(v)$, v is not blocked
	then	create new node w with $\mathcal{L}(\langle v, w\rangle) = \{R\}$ and $\mathcal{L}(w) = \{C\}$, and add $x_{w:C} \otimes x_{(v,w):R} \geq x_{v:\exists R.C}$ to $\mathcal{C}_{\mathcal{F}}$
$(\sqsubseteq)$	if	$\langle C \sqsubseteq D, n\rangle \in \mathcal{T}$, v is not indirectly blocked
	then	$\mathcal{L}(v) \to \mathcal{L}(v) \cup \{C, D\}$, and add $x_{v:D} \geq x_{v:C} \otimes n$ to $\mathcal{C}_{\mathcal{F}}$

Analytical Fuzzy Tableaux: $\mathcal{ALC}$ under SFL over $[0, 1]$

- Works as for classical $\mathcal{ALC}$ on completion forests
  - Node labels $\mathcal{L}(v)$ contain, rather than DL concept expressions, expressions of the form $\langle C, n\rangle$

    "The truth degree of being v instance of C is $\geq n$"

  - Blocking is as for classical $\mathcal{ALC}$
  - The completion forest is expanded by repeatedly applying inference rules
  - The completion-forest is complete when none of the rules are applicable
- Additionally, we adapt the notion of clash: a clash is either
  - $\langle \bot, n\rangle$ with $n > 0$; or
  - a pair $\langle C, n\rangle$ and $\langle \neg C, m\rangle$ with $n > 1 - m$
- Eventually, the initial KB is satisfiable if there is a clash-free complete completion forest

- ($\sqcap$). If *(i)* $\langle C_1 \sqcap C_2, n\rangle \in \mathcal{L}(v)$, *(ii)* $\{\langle C_1, n\rangle, \langle C_2, n\rangle\} \not\subseteq \mathcal{L}(v)$, and *(iii)* node v is not indirectly blocked, then add $\langle C_1, n\rangle$ and $\langle C_2, n\rangle$ to $\mathcal{L}(v)$.
- ($\sqcup$). If *(i)* $\langle C_1 \sqcup C_2, n\rangle \in \mathcal{L}(v)$, *(ii)* $\{\langle C_1, n\rangle, \langle C_2, n\rangle\} \cap \mathcal{L}(v) = \emptyset$, and *(iii)* node v is not indirectly blocked, then add some $\langle C, n\rangle \in \{\langle C_1, n\rangle, \langle C_2, n\rangle\}$ to $\mathcal{L}(v)$.
- ($\forall$). If *(i)* $\langle \forall R.C, n\rangle \in \mathcal{L}(v)$, *(ii)* $\langle R, m\rangle \in \mathcal{L}(\langle v, w\rangle)$ with $m > 1 - n$, *(iii)* $\langle C, n\rangle \notin \mathcal{L}(w)$, and *(iv)* node v is not indirectly blocked, then add $\langle C, n\rangle$ to $\mathcal{L}(w)$.
- ($\exists$). If *(i)* $\langle \exists R.C, n\rangle \in \mathcal{L}(v)$, *(ii)* there is no $\langle R, n_1\rangle \in \mathcal{L}(\langle v, w\rangle)$ with $\langle C, n_2\rangle \in \mathcal{L}(w)$ such that $\min(n_1, n_2) \geq n$, and *(iii)* node v is not blocked, then create a new node w, add $\langle R, n\rangle$ to $\mathcal{L}(\langle v, w\rangle)$ and add $\langle C, n\rangle$ to $\mathcal{L}(w)$.
- ($\sqsubseteq$). If *(i)* $\langle \top \sqsubseteq D, n\rangle \in \mathcal{T}$, *(ii)* $\langle D, n\rangle \notin \mathcal{L}(v)$, and *(iii)* node v is not indirectly blocked, then add $\langle D, n\rangle$ to $\mathcal{L}(v)$.

Non-Deterministic Analytic Fuzzy Tableaux

- It's a combination of the analogous method for fuzzy propositional logic and analytical fuzzy tableau
- Rule example:

($\sqcap$). If *(i)* $\langle C_1 \sqcap C_2, m \rangle \in \mathcal{L}(v)$, *(ii)* there are $m_1, m_2 \in L_n$ such that $m_1 \otimes m_2 = m$ with $\{\langle C_1, m_1 \rangle, \langle C_2, m_2 \rangle\} \not\subseteq \mathcal{L}(v)$, and *(iii)* node v is not indirectly blocked, then add $\langle C_1, m_1 \rangle$ and $\langle C_2, m_2 \rangle$ to $\mathcal{L}(v)$

Reduction to Classical DLs

- Same principle as for the reduction for propositional fuzzy logic
- Needs adaption to the DL constructs: e.g. $\exists, \forall$ and $\sqsubseteq$
- Examples of reduction rules for SFL:

$$
\begin{array}{rl}
\rho(A, \geq \gamma) = & A_{\geq \gamma} \\
\rho(C \sqcap D, \geq \gamma) = & \rho(C, \geq \gamma) \sqcap \rho(D, \geq \gamma) \\
\rho(C \sqcap D, \leq \gamma) = & \rho(C, \leq \gamma) \sqcup \rho(D, \leq \gamma) \\
\rho(\forall R.C, \geq \gamma) = & \forall \rho(R, > 1-\gamma).\rho(C, \geq \gamma) \\
\rho(\forall R.C, \leq \gamma) = & \exists \rho(R, \geq 1-\gamma).\rho(C, \leq \gamma) \\
\rho(\exists R.C, \geq \gamma) = & \exists \rho(R, \geq \gamma).\rho(C, \geq \gamma) \\
\rho(\exists R.C, \leq \gamma) = & \forall \rho(R, > \gamma).\rho(C, \leq \gamma) \\
\rho(R, \geq \gamma) = & R_{\geq \gamma} \\
\rho(\langle a{:}C, \gamma \rangle) = & \{a{:}\rho(C, \geq \gamma)\} \\
\rho(\langle C \sqsubseteq D, n \rangle) = & \bigcup_{\alpha \in \bar{N}_{+}^{\mathcal{K}}, \alpha \leq n} \{\rho(C, \geq \alpha) \sqsubseteq \rho(D, \geq \alpha)\}
\end{array}
$$

Computational Complexity

The bad news...undecidability!

Proposition

Assume that fuzzy GCIs are restricted to be classical, i.e. of the form $\langle \alpha, 1 \rangle$ only. Then for the following fuzzy DLs, the KB satisfiability problem is undecidable over $[0, 1]$:

1. $\mathcal{ELC}$ *with classical axioms only under Łukasiewicz logic and product logic;*
2. $\mathcal{ELC}$ *under any non Gödelt-norm* $\otimes$*;*
3. $\mathcal{ELC}$ *with concept assertions of the form* $\langle \alpha = n \rangle$ *only under any non Gödelt-norm* $\otimes$*;*
4. $\mathcal{AL}$ *with concept implication operator* $\rightarrow$ *and concept assertions of the form* $\langle \alpha = n \rangle$ *only under any non Gödelt-norm* $\otimes$*.*
5. $\mathcal{ELC}$ *under SFL with weighted sum constructor.*

Some decidability results..

Proposition

The KB satisfiability problem is decidable for

- $\mathcal{SROIQ}$ *under SFL over* $[0, 1]$ *and Gödel logic over* L_n
- $\mathcal{SROIN}$ *under Łukasiewicz logic over* L_n
- $\mathcal{SHI}$ *under any continuous t-norm over* L_n *without TBox*
- $\mathcal{ALC}$ *with concept implication operator* $\rightarrow$*, for any continuous t-norm over* $[0, 1]$ *with acyclicTBox*
- $\mathcal{SHIF}$ *with concept implication operator* $\rightarrow$*, for Łukasiewicz logic over* $[0, 1]$ *with acyclicTBox*
- $\mathcal{SI}$ *under any continuous t-norm over* $[0, 1]$ *without TBox*

Reasoners

Languages supported by fuzzy ontology reasoners:

Reasoner	Fuzzy DL	Logic	Degrees	Other constructors	GUI
fuzzyDL	$\mathcal{SHIF}(\mathbf{D})$	Z, Ł	General	Modifiers, rough, aggregation	•
Fire	$\mathcal{SHIN}$	Z	Numbers		•
FPLGERDS	$\mathcal{ALC}$	Ł	Numbers	Role negatio/top/bottom	
YADLR	$\mathcal{ALCOQ}$	Z, Ł	General	Local reflexivity	
DeLorean	$\mathcal{SROIQ}(\mathbf{D})$	Z, G	General	Modifiers, rough DL	•
GURDL	$\mathcal{ALC}$	General	Numbers		No
FRESG	$\mathcal{ALC}(\mathbf{D})$	Z	Numbers	Fuzzy datatype expressions	•
LiFR	*DLP* fragment	Z	Numbers	Weighted concepts	
SMT-based solver	$\mathcal{ALE}$	Π	No	No	
DLMedia	*DLR–Lite*	Z, G	Numbers	Image similarity	•
SoftFacts	*DLR–Lite*	Z, G	Numbers	Fuzzy datataypes	•
ONTOSEARCH2	$DL-Lite_R$	General	Numbers		•

Reasoning services offered by fuzzy ontology reasoners

Reasoner	CON	ENT	CSAT	SUB	IR	BDB	Other tasks	OPT
fuzzyDL	•	•	•	•	•	•	Defuzzification	•
Fire	•	•	•	•		•	Classification	•
FPLGERDS		•						
YADLR		Partial			•	Partial	Realisation	
DeLorean	•	•	•	•		•		•
GURDL	•	•		•				•
FRESG	•	•	•		•		Realisation	
LiFR		Partial	•	•		•		
SMT-based solver			•					
DLMedia							Top-k Image Retrieval	•
SoftFacts							Top-k CQA	•
ONTOSEARCH2							Retrieval	

“CON”, “ENT”, “CSAT”, “SUB”, “IR”, “BED”, and “OPT” represent consistency, entailment, concept satisfiability, subsumption, instance retrieval, BED, and optimisations, respectively

That's it !